Capital Accumulation and Limited Commitment*

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Abstract

We investigate under what condition the first-best allocation without commitment is sustainable in a production economy. We find that allowing capital accumulation can help to sustain the first-best allocation, although it is known to create a distortion. In an economy with productivity shocks, gains from efficient resource allocation between agents can be so large that it can compensate for the increase in the outside option that arises when capital moves to the more productive agent from the less productive agent.

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Key words: Limited Commitment, Enforcement Constraints, First-best Allocation, Production Economy

Introduction

We investigate properties of a production economy under limited commitment by focusing on the role of capital accumulation. It is widely accepted that allowing capital accumulation usually creates a distortion and thus makes the efficient allocation less sustainable in a production

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economy with limited commitment. For example, Kehoe and Perri (2004) show that once capital is introduced into their model, an additional capital income tax is required to decentralize a competitive equilibrium. Acemoglu, Golosov, and Tsyvinski (2008) also show that allowing capital accumulation creates additional distortion and hence generates a rationale for long-run capital taxation. Abraham and Carceles-Poveda (2006) reach a similar conclusion.

In this paper, however, we show that this prediction is not always true. We present a simple model incorporating incentives for risk sharing and capital reallocation, and an outside option for two agents. We derive the conditions under which the efficient allocation is sustainable in this economy and show that the first-best allocation becomes sustainable once we allow capital accumulation.

Our model is an extension of Kocherlakota (1996) into a production economy. There are two infinitely-lived agents endowed with the same decreasing returns to scale production technology. They are uncertain about their future productivity. The productivity shocks between the agents are negatively correlated, and we assume no aggregate uncertainty for simplicity. Agents are always better off ex-ante agreeing with the efficient allocation because they can insure each other and they can efficiently allocate capital so that the output from the contract is greater than the sum of each agent's output. However, in each period after the shock is realized, the agent with higher productivity may have an incentive to deviate from the contract not only because he has higher productivity but also because he is assigned a greater amount of capital to maximize aggregate production.

We analytically characterize the conditions under which the first-best allocation is sustainable with limited commitment in terms of parameters representing preference, uncertainty and technology. First, our results are consistent with the literature regarding the impact of preference and uncertainty. As agents value future consumption more or as the fluctuation of productivity shock rises, the first-best allocation is more likely to be sustainable due to the risk sharing motive.

More importantly, the implications of technology for sustainability are not straightforward in that they critically depend on the value of other parameters. We present interesting cases in which the first-best allocation, which is not sustainable in an endowment economy, is sustainable in a production economy. We identify that the driving force behind this result is capital accumulation: when capital accumulation is not allowed, the efficient allocation becomes unsustainable.

At first glance, this result might sound counter-intuitive because it is well known that accumulating more capital can make agents more able to self-insure against future shocks and hence increase the outside option value. However, in our setting, allowing capital accumulation actually resolves the existing distortion by increasing the value from the planner's problem more than the value from the outside option. We show that, for some technology, there is a large additional gain from efficient resource allocation by the planner in addition to the consumption risk sharing. This gain in efficiency can be enough to compensate for the increase in the outside option that arises when capital moves to the more productive agent from the less productive agent.

Our work is closely related to the literature on limited commitment. Following Kehoe and Levine (1993) and Kocherlakota (1996), these types of models have been widely used to understand asset pricing (e.g., Alvarez and Jermann, 2002; Azariadis and Cass, 2007), international finance (e.g., Kehoe and Perri, 2002; Aguiar, Amardor and Gopinath, 2009), firm dynamics (e.g., Albuquerque and Hopenhayn, 2004), consumption distribution (e.g., Krueger and Perri, 2005) and political economy (e.g., Acemoglu, Golosov and Tsyvinski, 2008). We study properties of production economy with limited commitment and show that allowing capital accumulation can resolve a distortion, which is a unique feature of our paper. In addition, while the inefficiency problem arising from capital accumulation or savings has received sufficient attention in the existing literature, the comparison of endowment to consumption-savings with limited commitment has not been explicitly investigated as in our work.

The model environment is described in Section 1. In Section 2 we specify the value from the planner's problem with commitment. Section 3 describes the limited commitment in our environment. In section 4, we impose a parametric assumption on our environment and analytically characterize the conditions under which the first-best allocation is sustainable. The implication of the key parameters on the conditions as well as the intuition behind the result are studied. The main result derived in Section 4 is verified in a more general stochastic environment in Section 5. In Section 6 we offer concluding remarks.

1 Environment

Preference The economy consists of two infinitely-lived agents indexed by $i \in \{1, 2\}$. They value consumption (c) in each period according to a utility function $u(.)$ which is a strictly concave, increasing and differentiable function with $u'(0) = \infty$.

Technology Each agent has decreasing returns to scale production technology given by

$$Y_t^i = A_t^i F(K_t^i) \quad \text{for} \quad t = 0, 1, 2, ...$$

Each agent is endowed with the same amount of capital at period 0. That is, we assume $K_0^1 = K_0^2$.

Uncertainty There are two states $s_t \in \{H, L\}$ with respect to the productivity A_t^i of each agent i in period t for $t = 1, 2, ...$, which follows a binary Markov process with transition probabilities given by

$$\pi(s|s') = \Pr(s_{t+1} = s' \,|\, s_t = s), \quad s, s' \in \{H, L\}.$$

Denote $s^t = (s_1, \ldots, s_t)$. We define $\pi(s^t)$ by the probability for s^t being realized. The productivity shock is symmetric. In other words, we have either $(A_t^1, A_t^2) = (1+\alpha, 1-\alpha)$ or $(1-\alpha, 1+\alpha)$, so that there is no aggregate productivity shock. At $t = 0$, the probability of $(A_1^1, A_1^2) = (1 + \alpha, 1 - \alpha)$ is $\frac{1}{2}$.

The agents are symmetric at period 0. Each agent is uncertain of the productivity in the next period. The assumption of the negative correlated business idea is a simple way of generating a situation in which agents are better off ex-ante when they collectively maximize their utility; it not only reduces consumption volatility, but also enlarges the production possibility frontier by allocating more resources to the agent who achieves a higher productivity. We assume that the planner allocates capital for each agent after the shock is realized given that the aggregate capital is allocated one period ahead.[1]

[1] We could have assumed that the capital for each agent is installed one period ahead. Our argument is more easily and clearly addressed with the current assumption.

2 Optimal Contract with Commitment

We first derive the optimal contract (the first-best allocation) between two agents without considering whether one of the agents can deviate from the contract at any given time. The planner's problem is to allocate $\{(c^1(s^t), c^2(s^t))\}_{t=1}^{\infty}, \{(K^1(s^t), K^2(s^t))\}_{t=0}^{\infty}$ in order to maximize

$$\max E_0 \left[\sum_{t=1}^{\infty} \beta^t \Big(u(c^1(s^t)) + u(c_t^2(s^t)) \Big) \right] = \sum_{t=1}^{\infty} \sum_{s_t} \beta^t \Big(u(c^1(s^t)) + u(c_t^2(s^t)) \Big) \pi(s^t | s^0)$$

subject to the resource constraint for each t and s^t,

$$(RC) \qquad \sum_{i=1,2} \{c^i(s^t) + K^i(s^t)\} = \sum_{i=1,2} \{A_{t-1}^i F(K^i(s^{t-1})) + (1-\delta) K^i(s^{t-1})\}$$

$$K^i(s^t) \geq 0, \quad \forall i$$

given $K(s^0) = K^1(s^0) + K^2(s^0)$.

The first-order conditions with the Lagrange multiplier $\beta^t \pi(s^t) \mu(s^t)$ attached to the resource constraint are given for each t and s^t,

$$u(c^i(s^t)) = \mu(s^t) \quad \forall i = 1, 2 \tag{2.1}$$

$$\mu(s^t)\pi(s^t) = \beta \sum_{s^{t+1}|s^t} \mu(s^{t+1}) \pi(s^{t+1}) [A_t^i F'(K^i(s^t))) + (1-\delta)]. \tag{2.2}$$

Hence

$$1 = \beta \sum_{s^{t+1}|s^t} \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} \pi(s^{t+1}|s^t) [A_t^i F'(K^i(s^t))) + (1-\delta)] \quad \forall i = 1, 2$$

Therefore, we have

$$E\left[m(s', s) A^1(s) F'(K^1(s)) | s \right] = E\left[m(s', s) A^2(s) F'(K^2(s)) | s \right]$$

or

$$A^1(s) F'(K^1(s)) = A^2(s) F'(K^2(s)) \tag{2.3}$$

where $m = m(s', s)$ is the pricing kernel as in the usual asset pricing literature:

$$m(s', s) = \frac{\beta u'(c(s'))}{u'(c(s))} \pi(s'|s).$$

Notice (2.1) tells that $c^1(s^t) = c^2(s^t)$ for all t. Therefore, the pricing kernel has no index i.

3 Limited Commitment and the Outside Option

This section considers limited commitment and the outside option value. In particular, we assume each agent can deviate from the optimal contract after the resource is allocated and before the next period shock is realized.

In order to set up the participation constraint for each agent, we first need to pin down the outside option value at the beginning of time t defined by $V_a^i(K^i(s^t))$. It is the value the agent will get from his own planning problem with the initial capital $K^i(s^t)$. We assume that once one of the agents deviates from the optimal contract, he cannot form a contract again. This assumption is rather restrictive but simplifies the analysis considerably.

Note that at the end of time period $t-1$ before a new state is realized and before he is given consumption $c^t(s^t)$, each agent is assigned $K^i(s^t)$ amount of capital. Then, the outside option value is the solution to the following Cass-Koopmans' problem with the initial condition, $k^i(s^t) = K^i(s^t)$[2]:

$$V_a(K^i(s^t)) = \max \sum_{j=t}^{\infty} \sum_{s^j|s^t} \beta^{j-t} u(c^i(s^j))\pi(s^j|s^t)$$

subject to

$$c^i(s^j) + k^i(s^{j+1}) = A_j^i F(k^i(s^j)) + (1-\delta)k^i(s^j)$$

$$\forall s^j|s^t,\ \forall j \geq t.$$

Notice that $\pi(s^j|s^t) = \Pr(s^j|s^t)$ for $j = t, t+1, t+2, \cdots$.

The following participation constraint should be satisfied by both agents in order for an agent not to deviate from the optimal contract. That is, for all $i = 1, 2$,

$$(PCi) \quad \sum_{j=t}^{\infty} \sum_{s^j|s^t} \beta^{j-t} u(c^i(s^j))\, \pi(s^j|s^t) \geq V_a^i(K^i(s^t)), \quad \forall s^t,\ \forall j \geq t.$$

The main focus of this paper is to find out when (PCi) is satisfied and when is not.

[2]In order to distinguish the allocation from the optimal contract, we use the lower case letter for the outside option value.

4 Deterministic Case

We first consider a deterministic case in which we can derive the analytic solution both for the value from the optimal contract and the value from outside option.

Assumption 1.

$$\pi(s^t = H \,|\, s^{t-1} = L) = \pi(s^t = L \,|\, s^{t-1} = H) = 1.$$

$$u(c) = \log(c), \quad \delta = 1, \quad A_t F(K_t) = A_t K_t^{\gamma} \quad \text{where} \quad \gamma \in [0, 1).$$

Proposition 1. *Under Assumption 1, the first-best allocation is sustainable if and only if*

$$H(\beta,\alpha,\gamma) := \log\Gamma(\alpha,\gamma) - \frac{(1-\gamma)\{\beta\log(1-\alpha)+\log(1+\alpha)\}}{(1-\beta\gamma)(1+\beta)} - \frac{(1-\beta)\gamma}{1-\beta\gamma}\log(1+\alpha) \geq 0 \tag{4.1}$$

where $\Gamma(\alpha,\gamma) = \left(\frac{(1+\alpha)^{\frac{1}{1-\gamma}}+(1-\alpha)^{\frac{1}{1-\gamma}}}{2}\right)^{1-\gamma}$.

Proof. See the Appendix. □

Note that the condition (4.1) is a function of (β, α, γ) so that we can define the left-hand-side of the inequality condition (4.1) by $H(\beta, \alpha, \gamma)$. We first investigate the relationship between β and H.

Corollary 1. *For any given* $\alpha \in [0, 1)$ *and* $\gamma \in [0, 1)$*, there exists the unique* $\hat{\beta}$ *such that for* $\beta > \hat{\beta}$ *the first best allocation is sustainable.*

Proof. It is easy to check, $H(0, \alpha, \gamma) < 0$, $H(1, \alpha, \gamma) > 0$, and $\frac{\partial^2 H(\beta,\alpha,\gamma)}{\partial\beta^2} > 0$ for any given $\alpha \in [0, 1)$ and $\gamma \in [0, 1)$. Therefore, there is the unique $\hat{\beta}$ such that the condition (4.1) is violated if $\beta < \hat{\beta}$ and (4.1) is satisfied if $\beta \geq \hat{\beta}$. □

With the same argument, we can prove the following corollary.

Corollary 2. *For any given* $\beta \in [0, 1)$ *and* $\gamma \in [0, 1)$*, there exists the unique* $\hat{\alpha}$ *such that for* $\alpha > \hat{\alpha}$ *the first best allocation is sustainable.*

Corollaries 1 and 2 confirm the rather well-known results in the literature. As the income fluctuation rises, and as agents value future consumption more, the optimal contract, by which agents can achieve risk sharing, is more likely to be sustainable.

The unknown result is the relationship between γ and the condition (4.1). We find that the effect of γ on the condition (4.1) is ambiguous in that it critically depends on the value of β and α. For example, $\frac{\partial H}{\partial \gamma}$ is highly nonlinear and depends on the value of β and α. Given β and α, as γ increases, the difference between the marginal productivity of capital of the two agents increases. This means that capital allocated to the agent receiving a positive shock should also increase to equalize the marginal productivity of capital (MPK) across agents. Since the value from outside option is strictly increasing in the initial capital as shown in the Appendix, the value from the outside option of the agent with a positive shock increases as γ increases. On the other hand, the increase in the difference between MPKs means that the amount of the rise in the aggregate output by optimally allocating capital across agents also increases. In other words, the value from the optimal contract increases as γ increases. Therefore, the overall effect of γ is nonlinear.

We present a particularly interesting example. Figure 1 plots $H(0.9, 0.1, \gamma)$ for $\gamma \in [0, 1)$. As shown in the figure, the condition (4.1) is satisfied for $\gamma \in [0.96, 1)$ and is violated otherwise. In this example, the first-best allocation is not sustainable for an endowment economy with $\gamma = 0$, but is sustainable in a production economy with a large γ.[3]

To emphasize the role of capital accumulation, we investigate the case where the agents are prevented from inter-temporal optimization (i.e., capital accumulation) in the following proposition.

Proposition 2. *Under Assumption 1, if the planner is prevented from engaging in capital accumulation, i.e., initial capital is given and there is no investment afterward, then the first-best allocation is sustainable if and only if*

$$\hat{H}(\beta, \alpha, \gamma) := \log \hat{\Omega} - \frac{\log(1+\alpha)}{1+\beta} - \frac{\beta \log(1-\alpha)}{1+\beta} - \gamma \log \pi \geq 0 \tag{4.2}$$

where $\hat{\Omega} = \frac{1}{2}\Big((1+\alpha)\pi^{\gamma} + (1-\alpha)(1-\pi)^{\gamma}\Big)$ *and* $\pi = \left\{1 + \left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1}{\gamma-1}}\right\}^{-1}$. *In addition, (4.2) is*

[3] See other examples and the descriptions for them in the Appendix. All numerical examples basically show that the relationship is nonlinear.

true regardless of whether capital depreciates over time or not.

Proof. See the Appendix. □

Define the left-hand-side of the inequality condition (4.2) by $\hat{H}(\beta, \alpha, \gamma)$. We plot $\hat{H}(0.9, 0.1, \gamma)$ for $\gamma \in [0, 1)$ in Figure 2, in which the first-best allocation is never sustainable for all $\gamma \in [0, 1)$, although the same parameter values are used as in Figure 1. If capital accumulation is prohibited, the planner only execute the static maximization, not the intertemporal optimization. There might be the gains from the optimal contract: By allocating capital properly, the aggregate production can be maximized for every period and each agent should forgo the present discounted value of future gains if he deviates from the contract. However, the present discounted value of the stream of future static gains is insufficient to make the optimal (static) allocation sustainable as shown by Figure 2. Notice that the gains from capital accumulation are captured by the difference between two graphs in Figures 1 and 2.[4]

Then, why is capital accumulation important in our setting? It is related to the fact that capital could be used to insure against future shock if one agent were to deviate from the optimal contract. Once deviating, the agent should over-accumulate capital as is well-known by the incomplete market risk sharing literature (e.g., Aiyagari 1994). However, the optimal contract already provides perfect risk sharing and thus capital is used only for optimal production and not for insurance. For some technologies (or for some γ values), there is a large additional gain from efficient resource allocation by the planner in addition to the consumption risk-sharing. This gain in efficiency can be enough to compensate for the increase in the outside option that arises when capital moves to one agent to the other.

We further investigate, given capital accumulation, why the optimal contract is sustainable for one technology but not for another kind. To do so, we consider the difference in the production possibility frontier between the planner's output and the sum of corresponding autarky output at each period. First, consider the economy with $(\beta = 0.9, \alpha = 0.1, \gamma = 0.5)$. As shown in Figure 1, the first-best allocation is not sustainable in this economy. Figure 3 draws the output path by autarky and by the optimal contract. Even if the outcome fluctuation (and hence the consumption fluctuation) is high in autarky, the agent chooses to deviate from

[4]Note that the scales in Figures 1 and 2 are different. The difference $H(0.9, 0.1, \gamma) - \hat{H}(0.9, 0.1, \gamma) > 0$ for any $\gamma > 0$.

the optimal contract. In contrast, Figure 4 considers the economy with ($\beta = 0.9, \alpha = 0.1, \gamma = 0.96$). This figure shows that the agent does not deviate from the optimal contract although the outcome fluctuation (and hence the consumption fluctuation) is relatively low in autarky. Note that the optimal contract in both cases achieve perfect risk sharing. The difference between the two cases shows that the optimal contract when $\gamma = 0.96$ generates greater aggregate production than the optimal contract when $\gamma = 0.5$. In other words, the production frontier for the optimal contract with ($\beta = 0.9, \alpha = 0.1, \gamma = 0.96$) is larger than that for the economy with ($\beta = 0.9, \alpha = 0.1, \gamma = 0.5$).

5 Stochastic Case

The planner can achieve perfect risk sharing with commitment so that the value from the first-best allocation in the stochastic environment is the same as in the deterministic case. Therefore, the difference in the stochastic environment is solely driven by the change in the values from the outside option. Suppose $\pi(s^t = H \,|s^{t-1} = L) = \pi(s^t = L \,|s^{t-1} = H) = \pi$. As the shock becomes more persistent, that is, as π approaches zero, the autarky value of the agent with a positive shock increases and hence the participation constraint is more easily violated. For example, when $\beta = 0.9$, $\alpha = 0.1$ and $\gamma = 0.96$, the sustainability condition is violated for π, which is less than 0.75, and is not violated for π, which is greater than or equal to 0.75. Other than this, the main results derived in Section 4 hold in a stochastic environment. The computational procedure is described in the Appendix.

6 Conclusion

We study the implications of limited commitment on the first-best allocation in a production economy. To this end, we present a model in which two infinitely-lived agents with the same decreasing returns to scale technology face a negatively correlated productivity shock. Under a parametric assumption, the economy is characterized by the subjective discount factor (β), the extent of productivity shock (α) and the curvature of production function (γ). We analytically derive the condition under which the first-best allocation is sustainable with limited commitment. We show that capital accumulation can help to support the first-best allocation.

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Appendix: Proofs, Computational Procedure for Stochastic Cases, and Figures

A Proof of Proposition 1

Let us redefine the production function as

$$Y_t = A_t K_{t-1}^{\gamma} = K_{t-1}^{\gamma} l_t^{1-\gamma} = \hat{F}(K_{t-1}, l_t)$$

where $l_t \in \{(1+\alpha)^{\frac{1}{1-\gamma}}, (1-\alpha)^{\frac{1}{1-\gamma}}\}$. Note that the production function is the same as before with a different notation. Let $\Omega(\alpha, \gamma)$ be $\{(1+\alpha)^{\frac{1}{1-\gamma}} + (1-\alpha)^{\frac{1}{1-\gamma}}\}/2$.

(2.3) implies that

$$\hat{F}_1(K_t^1, l_{t+1}^1) = \hat{F}_1(K_t^2, l_{t+1}^2),$$

By homogeneity we have $\frac{K_t^1}{l_{t+1}^1} = \frac{K_t^2}{l_{t+1}^2}$. Let us define $k_t^f = \frac{K_t^1}{l_{t+1}^1} = \frac{K_t^2}{l_{t+1}^2} = \frac{K_t^1 + K_t^2}{l_{t+1}^1 + l_{t+1}^2}$. Notice that $c_t^1 = c_t^2$, $l_t^1 + l_t^2 = 2\Omega$ and $k_t^f = \frac{K}{2\Omega}$. The resource constraint collapses to

$$c_t + k_t^f \Omega(\alpha, \gamma) = f(k_{t-1}^f)\Omega(\alpha, \gamma), \tag{A.1}$$

where $f(k) = \hat{F}(k, 1) = k^{\gamma}$. Hence, by the equal treatment condition the planner's problem is to maximize $\sum_{t=0}^{t} \beta^t u(c_t)$ subject to (A.1). This is the same as the classical Cass-Koopmans growth model. Therefore, the solution with full enforcement is easily given by

$$c_t^f = (1-\beta\gamma)\Omega(\alpha, \gamma) f(k_t^f) \quad \text{and} \quad k_{t+1}^f = \beta\gamma f(k_t^f).$$

Recursively, we have $k^f_{t+j} = (\beta\gamma)^{\frac{1-\gamma^j}{1-\gamma}}(k^f_t)^{\gamma^j}$. Therefore, the value at given capital k^f_t is

$$
\begin{aligned}
V(k^f_t) &= \sum_{j=0}^{\infty} \beta^j \log(c_{t+j}) \\
&= \sum_{j=0}^{\infty} \beta^j \log\Big((1-\beta\gamma)\Omega(\alpha,\gamma) f(k^f_{t+j})\Big) \\
&= \frac{\log\Big((1-\beta\gamma)\Omega(\alpha,\gamma)\Big)}{1-\beta} + \gamma \sum_{j=0}^{\infty} \beta^j \log(k^f_{t+j}) \\
&= \frac{\log\Big((1-\beta\gamma)\Omega(\alpha,\gamma)\Big)}{1-\beta} + \frac{\gamma\log(\beta\gamma)}{1-\gamma}\left(\frac{1}{1-\beta} - \frac{1}{1-\gamma\beta}\right) + \frac{\gamma\log k^f_t}{1-\beta\gamma} \\
&= \frac{1}{1-\beta}\left[\log\Big((1-\beta\gamma)\Omega(\alpha,\gamma)\Big) + \frac{\beta\gamma\log(\beta\gamma)}{1-\beta\gamma}\right] + \frac{\gamma\log k^f_t}{1-\beta\gamma}.
\end{aligned}
$$

Notice that $\pi_H = \pi_L = 0$ implies $V_a(K,L) < V_a(K,H)$ where $V_a(K,s)$ is the outside option value given capital K with state $s \in \{H, L\}$. (These values are derived below.) An agent will choose the autarky only when he receives a large endowment. Therefore, he will never deviate if the following is satisfied:

$$
V\left(\frac{K}{2\Omega}\right) \geq V_a\left(\frac{K(1+\alpha)^{\frac{1}{1-\gamma}}}{2\Omega}, H\right),
$$

The above inequality is equivalent to (4.1), which completes the proof.

B Autarky Value

Suppose an agent is given K unit of capital and wants to deviate. Then, the autarky values $(V_a(K,H), V_a(K,L))$ are given by the following Bellman equation.

$$
\begin{aligned}
V_a(K,L) &= \max_{K'=K'(L)} \log\Big((1-\alpha)K^\gamma - K'\Big) + \beta V_a(K',H) \\
V_a(K,H) &= \max_{K'=K'(H)} \log\Big((1+\alpha)K^\gamma - K'\Big) + \beta V_a(K',L)
\end{aligned} \tag{B.1}
$$

where $K'(s)$ is investment given the current state $s \in \{H, L\}$. Note that in the deterministic case we consider, $\pi(H|H) = \pi(L|L) = 0$. We proceed with the following guess and verify argument. Guess

$$
V_a(K,L) = a_L \log K + x \quad \text{and} \quad V_a(K,H) = a_H \log K + y.
$$

The first order condition and the envelop theorem give $a_H = a_L = \frac{\gamma}{1-\beta\gamma}$ and

$$K'(L) = \beta\gamma F(K, 1-\alpha) \quad \text{and} \quad K'(H) = \beta\gamma F(K, 1+\alpha). \tag{B.2}$$

Then, putting (B.2) into the Bellman equation (B.1), we can match the coefficients with respect to x and y to get the following linear equation:

$$\begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma) + \frac{1}{1-\beta\gamma}\log(1-\alpha) \\ \log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma) + \frac{1}{1-\beta\gamma}\log(1+\alpha) \end{bmatrix}$$

Then, we get

$$\begin{aligned} x =& \frac{1}{1-\beta}\left[\log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma)\right] + \frac{\log(1-\alpha) + \beta\log(1+\alpha)}{(1-\beta\gamma)(1-\beta)(1+\beta)} \\ y =& \frac{1}{1-\beta}\left[\log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma)\right] + \frac{\beta\log(1-\alpha) + \log(1+\alpha)}{(1-\beta\gamma)(1-\beta)(1+\beta)} \end{aligned}$$

which verifies the guess.

C Proof of Proposition 2

We first derive the autarky value. Without loss of generality, we only consider the agent with higher productivity in the period 0. Suppose the agent starts with k_0 amount of capital. Then, his consumption in each period is $c_0 = (1+\alpha)k_0^\gamma$, $c_1 = (1-\alpha)k_0^\gamma(1-\delta)^\gamma$, $c_2 = (1+\alpha)k_0^\gamma(1-\delta)^{2\gamma}$, and so on. Thus, the autarky value $V_a(k_0)$ is obtained as follows.

$$\begin{aligned} V_a(k_0) &= \sum_{t=0}^{\infty} \beta^t \log c_t \\ &= \frac{\log(1+\alpha)}{1-\beta^2} + \frac{\beta\log(1-\alpha)}{1-\beta^2} + \frac{\gamma}{1-\beta}\log k_0 + \frac{\gamma\beta}{(1-\beta)^2}\log(1-\delta) \end{aligned}$$

Given that initial aggregate capital is K_0, the amount of capital allocated to the high productivity agent at each time t is

$$x^* := \pi K_0 = \arg\max_x \left\{(1+\alpha)x^\gamma + (1-\alpha)(K_0 - x)^\gamma\right\},$$

where $\pi = \frac{1}{1+\left(\frac{1+\alpha}{1-\alpha}\right)^{\frac{1}{\gamma-1}}}$. Hence, we have $c_0 = \frac{1}{2}\Big((1+\alpha)\pi^\gamma + (1-\alpha)(1-\pi)^\gamma\Big)K_0^\gamma$, $c_1 = \frac{1}{2}\Big((1+\alpha)\pi^\gamma + (1-\alpha)(1-\pi)^\gamma\Big)K_0^\gamma(1-\delta)^\gamma$, and so on. Thus, an agent's value V from the first-best contract is

$$
\begin{aligned}
V &= \sum_{t=0}^{\infty} \beta^t \log c_t \\
&= \frac{\log \hat{\Omega}}{1-\beta} + \frac{\gamma \log K_0}{1-\beta} + \frac{\gamma\beta}{(1-\beta)^2} \log(1-\delta)
\end{aligned}
$$

$$
\text{where} \quad \hat{\Omega} = \frac{1}{2}\Big((1+\alpha)\pi^{\gamma} + (1-\alpha)(1-\pi)^{\gamma}\Big)
$$

Therefore, the first-best allocation is sustainable if and only if

$$
V - V_a(\pi K_0) \geq 0 \quad \Leftrightarrow \log \hat{\Omega} - \frac{\log(1+\alpha)}{1+\beta} - \frac{\beta \log(1-\alpha)}{1+\beta} - \gamma \log \pi \geq 0
$$

A bit of algebra shows that the above argument is still true if capital depreciates with any $\delta > 0$ rate (since each side in the above equation has the same term corresponding to the depreciation and these two terms are eventually canceled out.) This completes the proof.

D Computational Procedure for Stochastic Case

1. Derive the value function from the first-best allocation.

$$
V(K) = max_{\{c,K^1,K^2,K'\}} \log(c) + \beta V(K')
$$

subject to

$$
c = \frac{1}{2}\Big\{(1+\alpha)F(K^1) + (1-\alpha)F(K^2) + (1-\delta)K - K'\Big\}
$$

for $K = K^1 + K^2$.

2. Derive the value function for autarky.

$$
V_a(k,s) = max_{\{c,k\}} \log(c) + \beta E\left[V_a(k',s')|s\right]
$$

subject to

$$
c = F(k,s) + (1-\delta)k - k'
$$

3. Given K, K^i is given by the policy function derived at Step 1, $K^i = K^i(K)$. The first best allocation is sustainable if and only if

$$
V(K) \geq V_a(K^1(K), H)
$$

E Figures

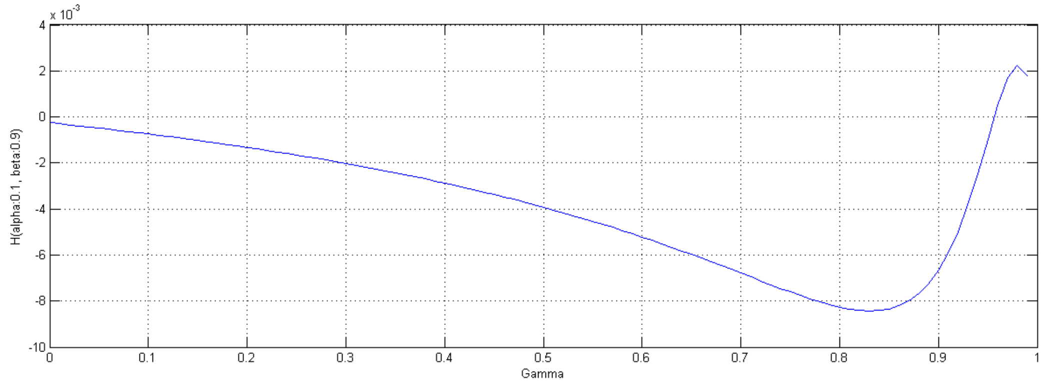

Figure 1: Y-axis:$H(\beta = 0.9, \alpha = 0.1, \gamma)$ X-axis:γ. Function H takes positive values for $\gamma \in [0, 1)$, in which the optimal contract is sustainable.

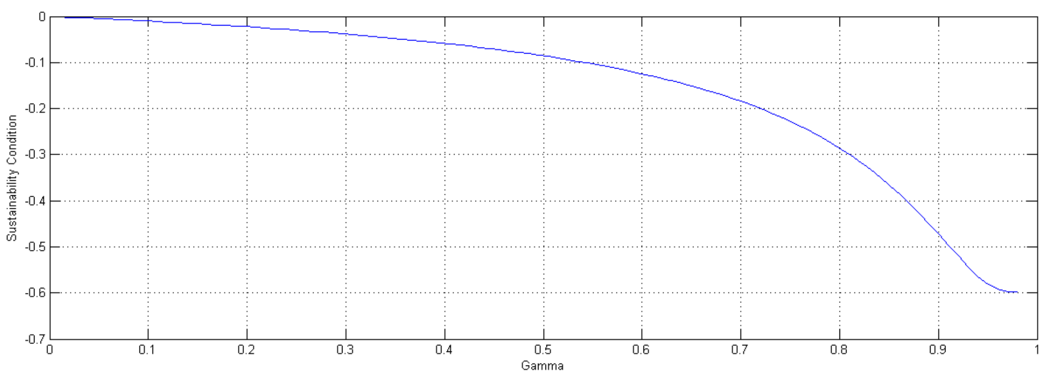

Figure 2: Y-axis:$\hat{H}(\beta = 0.9, \alpha = 0.1, \gamma)$ X-axis:γ. Function $\hat{H}$ is negative for any $\gamma \in [0, 1]$. Therefore, the optimal contract is never sustainable for any $\gamma \in [0, 1]$.

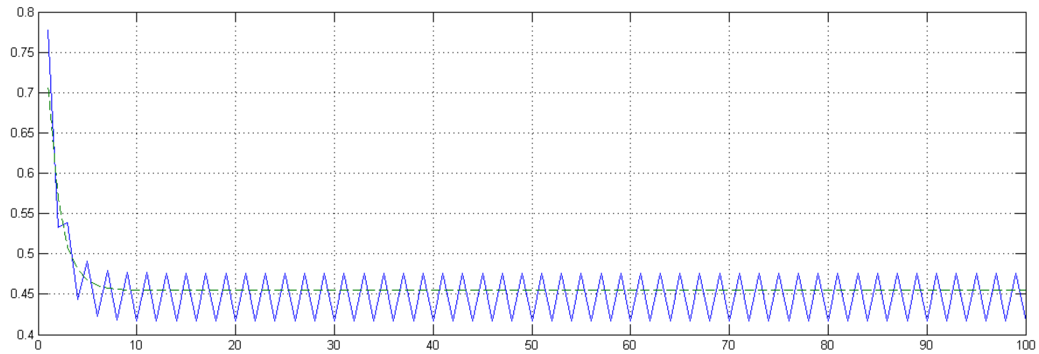

Figure 3: Time path for output *per capita* ($\beta = 0.9, \alpha = 0.1, \gamma = 0.5$) The blue line draws the outside option value with $k_0 = K_0/2$. The dotted green line draws the value from the optimal contract.

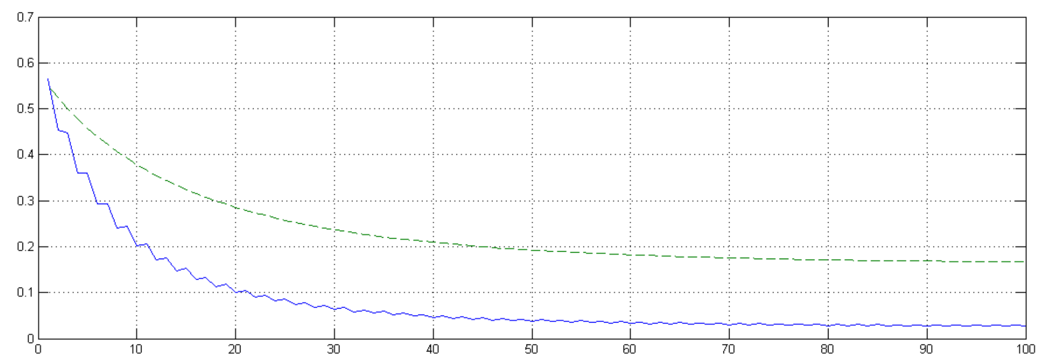

Figure 4: Time path for output *per capita* ($\beta = 0.9, \alpha = 0.1, \gamma = 0.96$) The blue line draws the outside option value with $k_0 = K_0/2$. The dotted green line draws the value from the optimal contract.

F Other Examples for $H(\beta, \alpha, \gamma)$

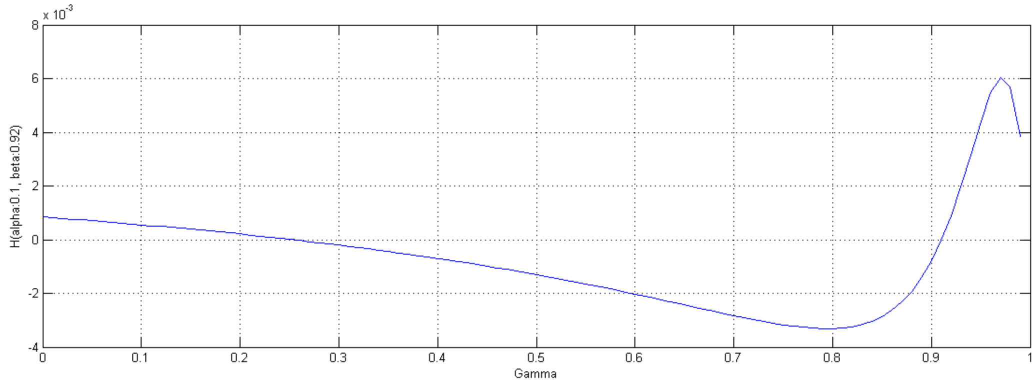

Figure 5: Y-axis:$H(\beta = 0.92, \alpha = 0.1, \gamma)$ X-axis:γ. The optimal contract is sustainable for $\gamma \in [0, 0.27)$ and $\gamma \in [0.91, 1)$.

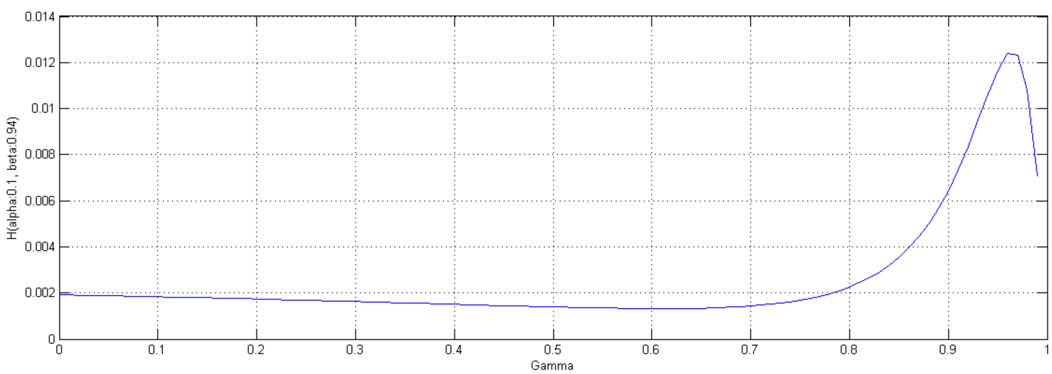

Figure 6: Y-axis:$H(\beta = 0.94, \alpha = 0.1, \gamma)$ X-axis:γ. The optimal contract is sustainable for all γ.

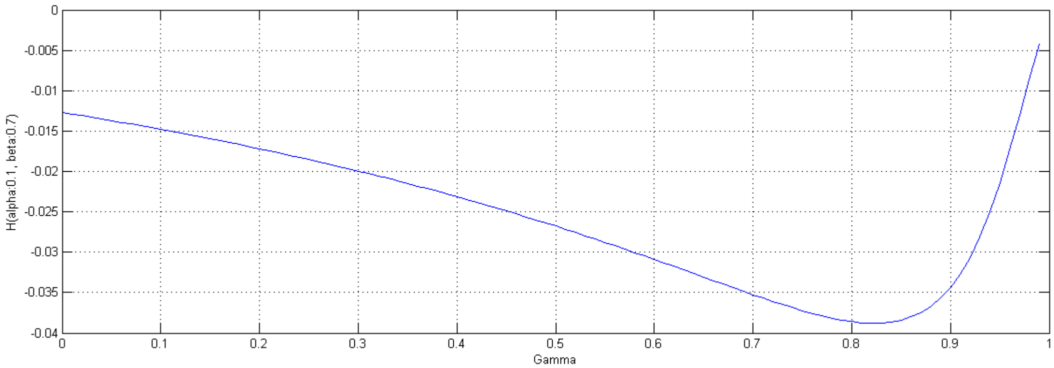

Figure 7: Y-axis:$H(\beta = 0.7, \alpha = 0.1, \gamma)$ X-axis:γ. The optimal contract is not sustainable for all γ.

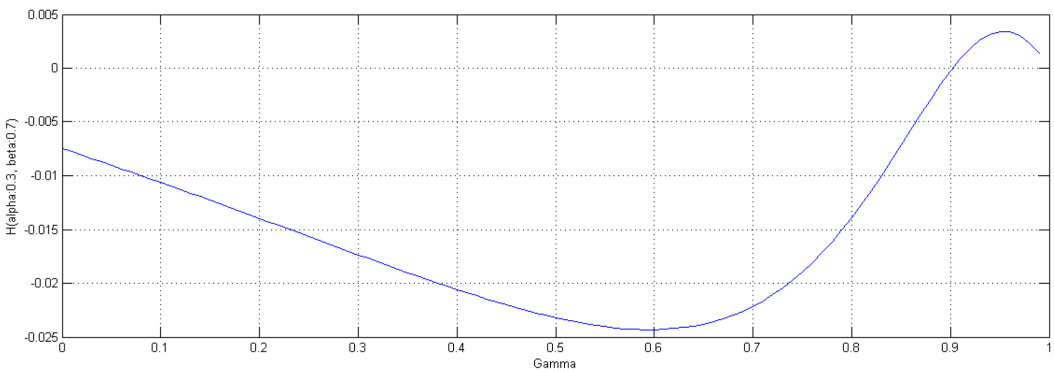

Figure 8: Y-axis:$H(\beta = 0.7, \alpha = 0.3, \gamma)$ X-axis:γ. The optimal contract is sustainable for $\gamma \in [0.90, 1)$.

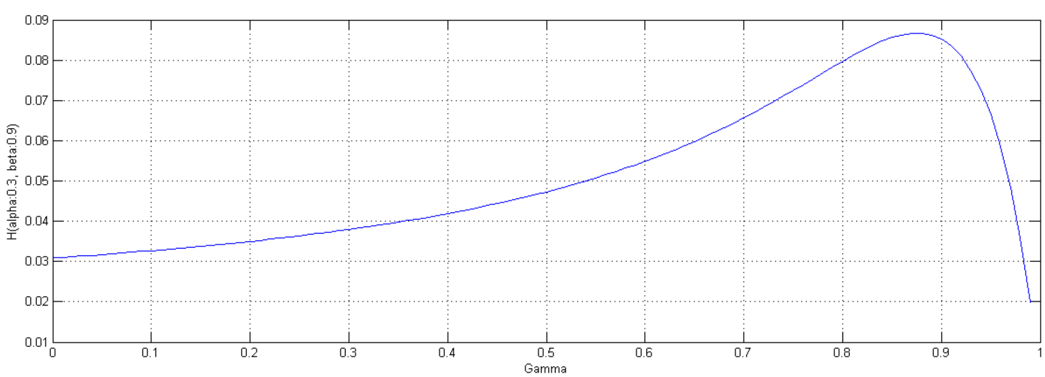

Figure 9: Y-axis:$H(\beta = 0.9, \alpha = 0.3, \gamma)$ X-axis:γ. The optimal contract is sustainable for all γ.