Capital Accumulation, Production Technology and Limited Commitment

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July 21, 2015

Abstract

We investigate the conditions under which the first-best allocation without commitment is sustainable in a production economy. We find that allowing capital accumulation can help to sustain the first-best allocation, although it is known to create a distortion. For some production technologies, gains from efficient resource allocation between agents can be so large that it can compensate for the increase in the outside option that arises when capital moves to the more productive agent from the less productive agent.

JEL classification number: D61, D90, E20, E22, E23

Key words: Limited Commitment, Enforcement Constraints, First-best Allocation, Production Economy

*We are grateful to Amitay Alter, Costas Azariadis, Robert Becker, George-Levi Gayle, Christian Hellwig, David Levine, Rody Manuelli, B. Ravikumar, Yongseok Shin, Ping Wang, and Stephen Williamson for their helpful comments and discussions on the current and the previous versions of this paper.

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Introduction

The implications of limited commitment on a transaction have been extensively studied in recent decades. The models with limited commitment have been used to understand asset pricing (e.g., Alvarez and Jerman (2000), Azariadisa and Kaas (2007)), international finance (e.g., Kehoe and Perri (2002), Aguiar et al (2008)), firm dynamics (e.g., Albuquerque and Hopenhayn (2004)), consumption distribution (e.g., Krueger and Perri (2006)) and political economy (e.g., Acemoglu et al. (2008)). In these types of economies, it is widely accepted that allowing capital accumulation usually creates a distortion since it increases the value from financial autarky and hence makes the enforcement constraint more likely be violated.

In this paper, we show that whether allowing capital accumulation creates additional distortion critically depends on production technology. For some technologies, gains from efficient resource allocation between agents can be so large that it can compensate for the increase in the outside option that arises when capital moves to the more productive agent from the less productive agent.

Our model is an extension of Kocherlakota (1996) into a production economy. There are two infinitely-lived agents endowed with the same decreasing returns to scale production technology. They are uncertain about their future productivity. The productivity shocks between the agents are negatively correlated, and for the sake of simplicity, we assume no aggregate uncertainty. Agents are always better off ex-ante agreeing with the efficient allocation because they can insure each other and because they can efficiently allocate capital so that the output from the contract is greater than the sum of each agent’s output. However, in each period after the shock is realized, the agent with higher productivity may have an incentive to
deviate from the contract not only because he has higher productivity but also because he is assigned a greater amount of capital to maximize aggregate production.

We characterize the analytic conditions under which the first-best allocation is sustainable with limited commitment in terms of preference, uncertainty and technology. First, our results are consistent with the literature regarding the impact of preference and uncertainty. As agents value future consumption more or as the fluctuation of productivity shock rises, the first-best allocation is more likely to be sustainable due to the risk-sharing motive. Second, and more importantly, the implications of technology for sustainability are not straightforward. For some technologies, allowing capital accumulation creates a distortion and hence the first-best allocation becomes unsustainable. However, for other technologies, the opposite is true: once capital is introduced and capital accumulation is allowed, the first-best allocation becomes sustainable.

At first glance, this result might seem counter-intuitive because it is well-known that adding capital and allowing capital accumulation increases the value from financial autarky and hence makes the enforcement constraint tighter. For example, Kehoe and Perri (2004) mentioned this effect as one of the main sources of distortions from adding capital and allowing capital accumulation. However, in our setting, allowing capital accumulation actually resolves the existing distortion by increasing the value from the planner’s problem more than the value from the outside option. We show that, for some technology, there is a large additional gain from efficient resource allocation by the planner in addition to the consumption risk-sharing. This gain in efficiency can be enough to compensate for the increase in the outside option that arises when capital moves to the more productive agent from the less productive agent.

A few papers have also studied properties of a production economy under limited com-
mitment. For example, Kehoe and Perri (2004) and Abraham and Carceles-Poveda (2006) focused on a decentralization of the constrained efficient allocation in a production economy without commitment. They show that with additional instrument such as capital income tax or an upper limit on the intermediaries capital holdings, the constrained efficient allocation can be decentralized when capital is introduced in an endowment economy. We take a different approach and provide a new insight into the properties of a production economy without commitment. Rather than characterizing the second-best allocation and its decentralization, we directly characterize the analytic condition for sustainability with respect to production technologies. We show that for some technologies, gains from efficient resource allocation between agents can be larger than the increase in the outside option of deviating from the efficient allocation.

The model environment is described in Section 1. In Section 2 we specify the value from the planner’s problem with commitment. Section 3 describes the limited commitment in our environment. In Section 4, we impose a parametric assumption on our environment and analytically characterize the conditions under which the first-best allocation is sustainable. We study the implication of the key parameters on the conditions, as well as the intuition behind the result. The main result derived in Section 4 is verified in a more general stochastic environment in Section 5. In Section 6 we offer concluding remarks.

1 Environment

Preference The economy consists of two infinitely-lived agents indexed by \( i \in \{1, 2\} \). They value consumption \((c)\) in each period according to a utility function \( u(.) \) which is a strictly
concave, increasing and differentiable function with $u'(0) = \infty$.

**Technology** Each agent has decreasing returns to scale production technology given by

$$Y^i_t = A^i_t F(K^i_t) \quad \text{for} \quad t = 0, 1, 2, ...$$

Each agent is endowed with the same amount of capital in period 0. That is, we assume $K^1_0 = K^2_0$.

**Uncertainty** There are two states $s_t \in \{H, L\}$ with respect to the productivity $A^i_t$ of each agent $i$ in period $t$ for $t = 1, 2, ..., $, which follows a binary Markov process with transition probabilities given by

$$\pi(s|s') = \Pr(s_{t+1} = s' | s_t = s), \quad s, s' \in \{H, L\}.$$ 

Denote $s^t = (s_1, \ldots, s_t)$. We define $\pi(s^t)$ by the probability of $s^t$ being realized. The productivity shock is symmetric. In other words, we have either $(A^1_t, A^2_t) = (1 + \alpha, 1 - \alpha)$ or $(1 - \alpha, 1 + \alpha)$, so that there is no aggregate productivity shock. At $t = 0$, the probability of $(A^1_1, A^2_1) = (1 + \alpha, 1 - \alpha)$ is $\frac{1}{2}$.

The agents are symmetric at period 0. Each agent is uncertain of the productivity in the next period. The assumption of the negative correlated business idea is a simple way of generating a situation in which agents are better off ex-ante when they collectively maximize their utility; it not only reduces consumption volatility, but also enlarges the production possibility frontier by allocating more resources to the agent who achieves higher productivity. We assume that the planner allocates capital for each agent after the shock is realized given that the aggregate capital is allocated one period ahead.$^1$

$^1$We could have assumed that the capital for each agent is installed one period ahead. Our argument is more easily and clearly addressed with the current assumption.
## 2 Optimal Contract with Commitment

We first derive the optimal contract (the first-best allocation) between two agents without considering whether one of the agents can deviate from the contract at any given time. The planner’s problem is to allocate \( \{(c^1(s^t), c^2(s^t))\}_{t=1}^{\infty}, \{(K^1(s^t), K^2(s^t))\}_{t=0}^{\infty} \) in order to maximize

\[
\max E_0 \left[ \sum_{t=1}^{\infty} \beta^t \left( u(c^1_t(s^t)) + u(c^2_t(s^t)) \right) \right] = \sum_{t=1}^{\infty} \sum_{s_t} \beta^t \left( u(c^1_t(s^t)) + u(c^2_t(s^t)) \right) \pi(s^t|s^0)
\]

subject to the resource constraint for each \( t \) and \( s^t \),

\[
(RC) \quad \sum_{i=1,2} \{c^i(s^t) + K^i(s^t)\} = \sum_{i=1,2} \{A^i_{t-1}F(K^i(s^{t-1})) + (1-\delta)K^i(s^{t-1})\}
\]

\[K^i(s^t) \geq 0, \quad \forall i\]

given \( K(s^0) = K^1(s^0) + K^2(s^0) \) where \( \beta \) is the discount factor.

The first-order conditions with the Lagrange multiplier \( \beta^t \pi(s^t) \mu(s^t) \) attached to the resource constraint are given for each \( t \) and \( s^t \),

\[
u(c^i(s^t)) = \mu(s^t) \quad \forall i = 1, 2 \quad (2.1)
\]

\[
u(s^t) \pi(s^t) = \beta \sum_{s^{t+1}|s^t} \mu(s^{t+1}) \pi(s^{t+1}) [A^i_t F'(K^i(s^t))] + (1-\delta)]. \quad (2.2)
\]

Hence

\[
1 = \beta \sum_{s^{t+1}|s^t} \frac{u'(c^i(s^{t+1}))}{u'(c^i(s^t))} \pi(s^{t+1}|s^t) [A^i_t F'(K^i(s^t))] + (1-\delta)] \quad \forall i = 1, 2
\]

Therefore, we have

\[
E \left[ m(s', s) A^1(s) F'(K^1(s)) | s \right] = E \left[ m(s', s) A^2(s) F'(K^2(s)) | s \right]
\]

or

\[
A^1(s) F'(K^1(s)) = A^2(s) F'(K^2(s)) \quad (2.3)
\]
where \( m = m(s', s) \) is the pricing kernel as in the usual asset pricing literature:

\[
m(s', s) = \frac{\beta u'(c(s'))}{u'(c(s))} \pi(s'|s).
\]

Notice (2.1) tells that \( c^1(s') = c^2(s') \) for all \( t \). Therefore, the pricing kernel has no index \( i \).

## 3 Limited Commitment and the Outside Option

This section considers limited commitment and the outside option value. In particular, we assume each agent can deviate from the optimal contract after the resource is allocated and before the next period shock is realized.

In order to set up the participation constraint for each agent, we first need to pin down the outside option value at the beginning of time \( t \) defined by \( V^t_a(K^i(s')) \). It is the value the agent will get from his own planning problem with the initial capital \( K^i(s') \). We assume that once one of the agents deviates from the optimal contract, he cannot form a contract again. This assumption is rather restrictive but simplifies the analysis considerably.

Note that at the end of time period \( t - 1 \) before he is given consumption \( c^t(s') \), each agent is assigned \( K^i(s') \) amount of capital. Then, the outside option value is the solution to the following Cass-Koopmans’ problem with the initial condition, \( k^i(s') = K^i(s')^2 \):

\[
V^t_a(K^i(s')) = \max_{j=t}^{\infty} \sum_{s|s'} \beta^{j-t} u(c^j(s')) \pi(s^j|s')
\]

subject to

\[
c^j(s') + k^i(s'^{j+1}) = A^j_i F(k^i(s^j)) + (1 - \delta) k^j(s^j)
\]

\(^2\)In order to distinguish the allocation from the optimal contract, we use the lower case letter for the outside option value.
\[ \forall s^j | s^t, \forall j \geq t. \]

Notice that \( \pi(s^j | s^t) = \Pr(s^j | s^t) \) for \( j = t, t + 1, t + 2, \ldots \).

The following participation constraint should be satisfied by both agents in order for an agent not to deviate from the optimal contract. That is, for all \( i = 1, 2 \),

\[
(PC_i) \sum_{j=t}^{\infty} \sum_{s^j | s^t} \beta^{j-t} u(c^i(s^j)) \pi(s^j | s^t) \geq V^i_a(K^i(s^t)), \quad \forall s^t, \forall j \geq t.
\]

The main focus of this paper is to find out when \((PC_i)\) is satisfied and when it is not.

### 4 Deterministic Case

We first consider a deterministic case in which we can derive the analytic solution for both the value from the optimal contract and the value from the outside option.

**Assumption 1.**

\[ \pi(s^t = H | s^{t-1} = L) = \pi(s^t = L | s^{t-1} = H) = 1. \]

\[ u(c) = \log(c), \quad \delta = 1, \quad A_t F(K_t) = A_t K_t^\gamma \quad \text{where} \quad \gamma \in [0, 1). \]

**Proposition 1.** Under Assumption 1, the first-best allocation is sustainable if and only if

\[
H(\beta, \alpha, \gamma) := \log \Gamma(\alpha, \gamma) - \frac{(1 - \gamma)(\beta \log(1 - \alpha) + \log(1 + \alpha))}{(1 - \beta \gamma)(1 + \beta)} - \frac{(1 - \beta) \gamma}{1 - \beta \gamma} \log(1 + \alpha) \geq 0
\]

(4.1)

where \( \Gamma(\alpha, \gamma) = \left( \frac{(1 + \alpha)^{\frac{1}{1 - \gamma}} + (1 - \alpha)^{\frac{1}{1 - \gamma}}}{2} \right)^{1 - \gamma}. \)

**Proof.** See the Appendix.
Note that the condition (4.1) is a function of \((\beta, \alpha, \gamma)\) so that we can define the left-hand-side of the inequality condition (4.1) by \(H(\beta, \alpha, \gamma)\). We first investigate the relationship between \(\beta\) and \(H\).

**Corollary 1.** For any given \(\alpha \in [0, 1)\) and \(\gamma \in [0, 1)\), there exists a unique \(\hat{\beta}\) such that for \(\beta > \hat{\beta}\) the first best allocation is sustainable.

*Proof.* It is easy to verify, \(H(0, \alpha, \gamma) < 0, H(1, \alpha, \gamma) > 0, \) and \(\frac{\partial^2 H(\beta, \alpha, \gamma)}{\partial \gamma^2} > 0\) for any given \(\alpha \in [0, 1)\) and \(\gamma \in [0, 1)\). Therefore, there is a unique \(\hat{\beta}\) such that the condition (4.1) is violated if \(\beta < \hat{\beta}\) and (4.1) is satisfied if \(\beta \geq \hat{\beta}\).

With the same argument, we can prove the following corollary.

**Corollary 2.** For any given \(\beta \in (0, 1)\) and \(\gamma \in [0, 1)\), there exists a unique \(\hat{\alpha}\) such that for \(\alpha > \hat{\alpha}\) the first best allocation is sustainable.

Corollaries 1 and 2 confirm the rather well-known results in the literature. As the income fluctuation rises, and as agents value future consumption more, the optimal contract, by which agents can achieve risk sharing, is more likely to be sustainable.

The unknown result is the relationship between \(\gamma\) and the condition (4.1). We find that the effect of \(\gamma\) on the condition (4.1) is ambiguous in that it critically depends on the value of \(\beta\) and \(\alpha\). For example, \(\frac{\partial H}{\partial \gamma}\) is highly nonlinear and depends on the value of \(\beta\) and \(\alpha\). Given \(\beta\) and \(\alpha\), as \(\gamma\) increases, the difference between the marginal productivity of capital of the two agents increases. This means that capital allocated to the agent receiving a positive shock should also increase to equalize the marginal productivity of capital (MPK) across agents. Since the value from outside option is strictly increasing in the initial capital as shown in the Appendix, the value from the outside option of the agent with a positive shock increases as \(\gamma\) increases.
On the other hand, the increase in the difference between MPKs means that the amount of
the rise in the aggregate output by optimally allocating capital across agents also increases.
In other words, the value from the optimal contract increases as $\gamma$ increases. Therefore, the
overall effect of $\gamma$ is ambiguous.

We present a particularly interesting example. Figure 1 plots $H(0.9, 0.1, \gamma)$ for $\gamma \in [0, 1)$. As shown in the figure, the condition (4.1) is satisfied for $\gamma \in [0.96, 1)$ and is otherwise violated. In this example, the first-best allocation is not sustainable for an endowment economy with $\gamma = 0$, but is sustainable in a production economy with a large $\gamma$.\(^3\)

We further investigate, given capital accumulation, why the optimal contract is sustainable for one kind of technology but not for another. To do so, we consider the difference in the production possibility frontier between the planner’s output and the sum of corresponding autarky output at each period. First, consider the economy with $(\beta = 0.9, \alpha = 0.1, \gamma = 0.5)$. As shown in Figure 1, the first-best allocation is not sustainable in this economy. Figure 2 draws the output path by autarky and by the optimal contract. Even if the outcome fluctuation (and hence the consumption fluctuation) is high in autarky, the agent chooses to deviate from the optimal contract. In contrast, Figure 3 considers the economy with $(\beta = 0.9, \alpha = 0.1, \gamma = 0.96)$. This figure shows that the agent does not deviate from the optimal contract although the outcome fluctuation (and hence the consumption fluctuation) is relatively low in autarky. Note that the optimal contract in both cases achieve perfect risk sharing. The difference between the two cases shows that the optimal contract when $\gamma = 0.96$ generates greater aggregate production than the optimal contract when $\gamma = 0.5$. In other words, the production frontier

\(^3\)See other examples and the descriptions of them in Appendix E. All numerical examples basically show that the relationship is nonlinear.
for the optimal contract with \((\beta = 0.9, \alpha = 0.1, \gamma = 0.96)\) is larger than that for the economy with \((\beta = 0.9, \alpha = 0.1, \gamma = 0.5)\).

5 Stochastic Case

We assumed there was no aggregate uncertainty. Therefore, the planner can achieve perfect risk sharing with commitment. The value from the first-best allocation in the stochastic environment is the same as in the deterministic case. The difference in the stochastic environment is solely driven by the change in the values from the outside option. Suppose \(\pi(s^t = H | s^{t-1} = L) = \pi(s^t = L | s^{t-1} = H) = \pi\). As the shock becomes more persistent, that is, as \(\pi\) approaches zero, the autarky value of the agent with a positive shock increases and hence the participation constraint is more easily violated. For example, when \(\beta = 0.9, \alpha = 0.1\) and \(\gamma = 0.96\), the sustainability condition is violated for \(\pi\), which is less than 0.75, and is not violated for \(\pi\), which is greater than or equal to 0.75. Other than this, the main results derived in Section 4 hold in a stochastic environment. The computational procedure is described in the Appendix.

6 Conclusion

We study the implications of limited commitment for the first-best allocation in a production economy. To this end, we present a model in which two infinitely-lived agents with the same decreasing returns to scale technology face a negatively correlated productivity shock. Under a parametric assumption, the economy is characterized by the subjective discount factor \((\beta)\), the extent of productivity shock \((\alpha)\) and the curvature of the production function \((\gamma)\). We
derive the analytic condition under which the first-best allocation is sustainable with limited commitment. We show that gains from efficient resource allocation between agents can be so large that it can compensate for the increase in the outside option that arises when capital moves to the more productive agent from the less productive agent.

References


Appendix: Proofs, Computational Procedure for Stochastic Cases, and Figures

A Proof of Proposition 1

Let us redefine the production function as

\[ Y_t = A_t K_t^\gamma = K_t^{\gamma} l_t^{1-\gamma} = \hat{F}(K_t, l_t) \]

where \( l_t \in \{(1 + \alpha)^{\frac{1}{\gamma}}, (1 - \alpha)^{\frac{1}{\gamma}}\} \). Note that the production function is the same as before with a different notation. Let \( \Omega(\alpha, \gamma) \) be \( \{(1 + \alpha)^{\frac{1}{\gamma}} + (1 - \alpha)^{\frac{1}{\gamma}}\}/2 \).

(2.3) implies that

\[ \hat{F}_1(K_t^1, l_t^1) = \hat{F}_1(K_t^2, l_t^2), \]

By homogeneity we have \( \frac{K_t^1}{l_t^1} = \frac{K_t^2}{l_t^2} \). Let us define \( k_t^f = \frac{K_t^1}{l_t^1} = \frac{K_t^2}{l_t^2} = \frac{K_t^1 + K_t^2}{l_t^1 + l_t^2} \). Notice that \( c_t^1 = c_t^2 \), \( l_t^1 + l_t^2 = 2\Omega \) and \( k_t^f = \frac{K_t}{2\Omega} \). The resource constraint collapses to

\[ c_t + k_t^f \Omega(\alpha, \gamma) = f(k_t^f)\Omega(\alpha, \gamma), \quad (A.1) \]
where \( f(k) = \hat{F}(k, 1) = k^\gamma \). Hence, by the equal treatment condition the planner’s problem is to maximize \( \sum_{t=0}^{T} \beta^t u(c_t) \) subject to (A.1). This is the same as the classical Cass-Koopmans growth model. Therefore, the solution with full enforcement is easily given by

\[
    c_t^f = (1 - \beta \gamma) \Omega(\alpha, \gamma) f(k_t^f) \quad \text{and} \quad k_{t+1}^f = \beta \gamma f(k_t^f).
\]

Recursively, we have \( k_{t+j}^f = (\beta \gamma)^{\frac{j}{1-\gamma}} (k_t^f)^{\gamma^j} \). Therefore, the value at given capital \( k_t^f \) is

\[
    V(k_t^f) = \sum_{j=0}^{\infty} \beta^j \log(c_{t+j}) = \sum_{j=0}^{\infty} \beta^j \log \left( (1 - \beta \gamma) \Omega(\alpha, \gamma) f(k_{t+j}^f) \right)
    = \log \left( (1 - \beta \gamma) \Omega(\alpha, \gamma) \right) \frac{1}{1-\beta} + \gamma \sum_{j=0}^{\infty} \beta^j \log(k_{t+j}^f)
    = \log \left( (1 - \beta \gamma) \Omega(\alpha, \gamma) \right) \frac{1}{1-\beta} + \gamma \log(\beta \gamma) \frac{1-\gamma}{1-\beta} - \frac{1}{1-\gamma} + \gamma \log k_t^f \frac{1}{1-\gamma}.
\]

Notice that \( \pi_H = \pi_L = 0 \) implies \( V_a(K, L) < V_a(K, H) \) where \( V_a(K, s) \) is the outside option value given capital \( K \) with state \( s \in \{ H, L \} \). (These values are derived below.) An agent will choose autarky only when he receives a large endowment. Therefore, he will never deviate if the following is satisfied:

\[
    V(\frac{K}{2\Omega}) \geq V_a(\frac{K(1 + \alpha)^{-\frac{1}{1-\gamma}}}{2\Omega}, H),
\]

The above inequality is equivalent to (4.1), which completes the proof.
B  Autarky Value

Suppose an agent is given $K$ unit of capital and wants to deviate. Then, the autarky values $(V_a(K, H), V_a(K, L))$ are given by the following Bellman equation.

$$
V_a(K, L) = \max_{K'=K'(L)} \log \left( (1 - \alpha) K^\gamma - K' \right) + \beta V_a(K', H)
$$

$$
V_a(K, H) = \max_{K'=K'(H)} \log \left( (1 + \alpha) K^\gamma - K' \right) + \beta V_a(K', L)
$$

(B.1)

where $K'(s)$ is investment given the current state $s \in \{H, L\}$. Note that in the deterministic case we consider, $\pi(H|H) = \pi(L|L) = 0$. We proceed with the following guess and verify the argument. Guess that

$$
V_a(K, L) = a_L \log K + x \quad \text{and} \quad V_a(K, H) = a_H \log K + y.
$$

The first order condition and the envelop theorem give $a_H = a_L = \frac{\gamma}{1 - \beta \gamma}$ and

$$
K'(L) = \beta \gamma F(K, 1 - \alpha) \quad \text{and} \quad K'(H) = \beta \gamma F(K, 1 + \alpha).
$$

(B.2)

Then, putting (B.2) into the Bellman equation (B.1), we can match the coefficients with respect to $x$ and $y$ to get the following linear equation:

$$
\begin{bmatrix}
1 & -\beta \\
-\beta & 1
\end{bmatrix}
\begin{bmatrix}
x \\
y
\end{bmatrix}
= \begin{bmatrix}
\log(1 - \beta \gamma) + \frac{\beta \gamma}{1 - \beta \gamma} \log(\beta \gamma) + \frac{1}{1 - \beta \gamma} \log(1 - \alpha) \\
\log(1 - \beta \gamma) + \frac{\beta \gamma}{1 - \beta \gamma} \log(\beta \gamma) + \frac{1}{1 - \beta \gamma} \log(1 + \alpha)
\end{bmatrix}
$$

Then, we get

$$
x = \frac{1}{1 - \beta} \left[ \log(1 - \beta \gamma) + \frac{\beta \gamma}{1 - \beta \gamma} \log(\beta \gamma) \right] + \frac{\log(1 - \alpha) + \beta \log(1 + \alpha)}{(1 - \beta \gamma)(1 - \beta)(1 + \beta)}
$$

$$
y = \frac{1}{1 - \beta} \left[ \log(1 - \beta \gamma) + \frac{\beta \gamma}{1 - \beta \gamma} \log(\beta \gamma) \right] + \frac{\beta \log(1 - \alpha) + \log(1 + \alpha)}{(1 - \beta \gamma)(1 - \beta)(1 + \beta)}
$$

which verifies the guess.
C  Computational Procedure for Stochastic Case

1. Derive the value function from the first-best allocation.

\[ V(K) = \max_{c,K^1,K^2,K'} [c, \log(c) + \beta V(K')] \]

subject to

\[ c = \frac{1}{2} \left\{ (1 + \alpha) F(K^1) + (1 - \alpha) F(K^2) + (1 - \delta) K - K' \right\} \]

for \( K = K^1 + K^2 \).

2. Derive the value function for autarky.

\[ V_a(k, s) = \max_{c,k} [c, \log(c) + \beta E[V_a(k', s')|s]] \]

subject to

\[ c = F(k, s) + (1 - \delta) k - k' \]

3. Given \( K, K^i \) is given by the policy function derived at Step 1, \( K^i = K^i(K) \). The first-best allocation is sustainable if and only if

\[ V(K) \geq V_a(K^1(K), H) \]
D Figures

Figure 1: Y-axis: $H(\beta = 0.9, \alpha = 0.1, \gamma)$, X-axis: $\gamma$

NOTE: Function $H$ takes positive values for $\gamma \in [0.96, 1)$, in which the optimal contract is sustainable.
Figure 2: Time path for output *per capita* ($\beta = 0.9, \alpha = 0.1, \gamma = 0.5, K_0 = 0.1$)

NOTE: The blue line draws the outside option value with $k_0 = K_0/2$. The dotted green line draws the value from the optimal contract.

Figure 3: Time path for output *per capita* ($\beta = 0.9, \alpha = 0.1, \gamma = 0.96, K_0 = 0.1$)

NOTE: The blue line draws the outside option value with $k_0 = K_0/2$. The dotted green line draws the value from the optimal contract.
Other Examples for $H(\beta, \alpha, \gamma)$

Figure 4: Y-axis: $H(\beta = 0.92, \alpha = 0.1, \gamma)$, X-axis: $\gamma$

NOTE: The optimal contract is sustainable for $\gamma \in [0, 0.27)$ and $\gamma \in [0.91, 1)$.

Figure 5: Y-axis: $H(\beta = 0.94, \alpha = 0.1, \gamma)$, X-axis: $\gamma$

NOTE: The optimal contract is sustainable for all $\gamma$. 
Figure 6: Y-axis: $H(\beta = 0.7, \alpha = 0.1, \gamma)$, X-axis: $\gamma$

NOTE: The optimal contract is not sustainable for all $\gamma$.

Figure 7: Y-axis: $H(\beta = 0.7, \alpha = 0.3, \gamma)$, X-axis: $\gamma$

NOTE: The optimal contract is sustainable for $\gamma \in [0.90, 1)$.

Figure 8: Y-axis: $H(\beta = 0.9, \alpha = 0.3, \gamma)$, X-axis: $\gamma$

NOTE: The optimal contract is sustainable for all $\gamma$. 

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