When Does Limited Commitment Matter in a Production Economy?*

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Abstract

We investigate the conditions under which the first-best allocation without commitment is sustainable in a production economy. While it is widely known in the literature that allowing capital accumulation creates a distortion, we find that it can help to sustain the first-best allocation. We also find that for a certain set of endowment economies in which the efficient allocation is not sustainable, the efficient allocation becomes sustainable once we introduce a production technology with small returns to scale. In some cases, gains from efficient resource allocation between agents can be so large that they can compensate for the increase in the outside option that arises when capital moves from the less-productive to the more-productive agent.

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Introduction

In many credit arrangements, a lender cannot force a borrower to commit to a contract to which both the lender and the borrower initially agreed. This problem, called the limited commitment, can distort risk-sharing among agents and lead to a welfare loss. It is well documented that in an endowment economy with limited commitment, the first-best allocation is more likely to be sustainable as the time discount factor or the variance of endowment increases. This is because, in such conditions, the value from the outside option decreases, and, as a result, agents tend to choose not to deviate from the optimal contract.

It is, however, not well understood how agents' default incentives change when a production technology is introduced into an economy. Under what conditions is the first-best allocation without commitment sustainable in a production economy? In particular, what role does a production technology play in achieving the first-best allocation? Understanding agents' default incentives is particularly relevant in a production economy because limited commitment can not only distort risk-sharing but also hinder resource allocation, which maximizes the aggregate production.

To answer these questions, we introduce a standard growth model featuring a risk-sharing problem between two agents under a decreasing returns to scale production technology. We then analytically characterize the conditions under which efficient resource allocation is sustainable without commitment. We find that the relationship between the degree of returns to scale technology and the agents' default incentives is highly non-linear. On the one hand, introducing a production technology and allowing capital accumulation can increase the value from financial autarky and, hence make it more likely that the enforcement constraint will be violated. On the other hand, allowing capital accumulation can increase the value from the optimal contract and, thus *help* to sustain the efficient resource allocation. In some cases, gains from efficient resource allocation between agents can be so large that they can compensate for the increase in the outside option that arises when capital moves from the less-productive to the more-productive agent. We also show that for a certain set of endowment economies in which the optimal contract is not sustainable, the efficient allocation becomes sustainable once we introduce a production technology with very small returns to scale or any returns to scale higher than the threshold.

Our result provides a new insight into the role of capital accumulation in agents' default incentives. It is well known, though not explicitly shown in the literature, that adding capital and allowing capital accumulation increase the value from financial autarky, thus making the enforcement constraint tighter (e.g., Kehoe and Perri (2004); Abraham and Carceles-Poveda (2006)). Our analytic characterization enables us to clearly show that the opposite can be true when there is a large additional gain from efficient resource allocation by the planner in addition to the consumption risk-sharing.

Our model features two types of infinitely-lived agents endowed with the same decreasing returns to scale production technology, both of whom are uncertain about their future productivity. The productivity shocks between the agents are negatively correlated, and for the sake of simplicity, we assume no aggregate uncertainty. Agents are always better off ex-ante agreeing with the efficient allocation because they can insure each other and because they can efficiently allocate capital so that the output from the contract is greater than the sum of both agents' output. However, in each period after the shock is realized, the agent with higher productivity may have an incentive to deviate from the contract, not only because he has higher productivity, but also because he is assigned a greater amount of capital to maximize aggregate production.

We characterize the analytic conditions under which the first-best allocation is sustainable with limited commitment in terms of the following three key parameters: the degree of uncertainty (α); time preference (β); and the degree of returns to scale (γ). We specify the benchmark endowment economy such that both the value function from the optimal contract and the value from the outside option in a production economy with $\gamma > 0$ uniformly converge to those in the benchmark endowment economy as γ goes to zero. Consistent with our specification, the sustainability condition derived in a production economy also converges to the condition separately derived in the benchmark endowment economy.

We confirm that our results with respect to the first two parameters are consistent with the literature on risk-sharing in an exchange economy (e.g., Kocherlakota (1996); Alvarez and Jerman (2000)): As agents value future consumption more or as the fluctuation of the productivity shock rises, the first-best allocation is more likely to be sustainable due to the risk-sharing motive. This consistency confirms the validity of our analytic characterization and also enables us to compare the exchange economy and the (production) economy in which capital accumulation is allowed.

Our results regarding the technology (that is, the degree of returns to scale) are novel in the literature. We first show that the relationship between the degree of returns to scale and the sustainability condition is non-monotonic. We also show that for a certain set of endowment economies in which the optimal contract is not sustainable, introducing a production technology with even extremely low returns to scale can improve risk-sharing, in the sense that the distance between the value from the optimal contract and the value from the outside option becomes smaller. More importantly, we find that for a certain set of endowment economies in which the efficient allocation is not sustainable, it becomes sustainable once we introduce a production technology with small returns to scale. In general, the statement holds with any returns to scale higher than the threshold.

The implications of a limited commitment to a transaction have been studied extensively in recent decades.[1] With a growing interest in models with limited commitment, a few papers

[1] The models with limited commitment have been used to understand asset pricing (e.g., Alvarez and Jerman (2000); Azariadisa and Kaas (2007)); international finance (e.g., Kehoe and Perri (2002); Aguiar et al (2008));

theoretically investigate the properties of a production economy under limited commitment. For example, Kehoe and Perri (2004) and Abraham and Carceles-Poveda (2006) focus on the decentralization of the constrained efficient allocation in a production economy without commitment. They show that with additional instruments such as capital income tax or an upper limit on the intermediaries capital holdings, the constrained efficient allocation can be decentralized when capital is introduced in an endowment economy. Rather than characterizing the second-best allocation and its decentralization, we directly characterize the analytic condition for sustainability with respect to production technologies and provide a new insight into the properties of a production economy without commitment. There are a few papers that discuss risk sharing with intertemporal technologies. Krueger and Perri (2006) introduce a productive tree in a Lucas tree economy and show that the higher is the return on this tree the more risk sharing can be achieved. Abraham and Laczo (2016) endogenize the size of the tree (public storage technology) and obtain a non-monotonic result between the return and the degree of risk sharing. Both papers introduce capital income in the Lucas-tree framework and their results have similar flavour with ours. Our paper establishes a clear link between the endowment economy and the production economy in terms of risk sharing.

The remainder of the paper is organized as follows. The model environment is described in Section 1. In Section 2, we specify the value from the planner's problem with commitment. Section 3 describes the limited commitment in our environment. In Section 4, we analytically characterize the conditions under which the first-best allocation is sustainable for a deterministic case; and we study the implications of the key parameters on the conditions, as well as the intuition behind the result. The main result derived in Section 4 is verified in a more general stochastic environment in Section 5. In Section 6, we offer concluding remarks.

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firm dynamics (e.g., Albuquerque and Hopenhayn (2004)); consumption distribution (e.g., Krueger and Perri (2006)); and political economy (e.g., Acemoglu et al. (2008)).

1 Environment

Preference The economy consists of two infinitely-lived agents indexed by $i \in \{1, 2\}$. They value consumption (c) in each period according to a utility function $u(.)$ which is a strictly concave, increasing and differentiable function with $u'(0) = \infty$.

Technology Each agent has decreasing returns to scale production technology given by

$$Y_t^i = A_t^i F(K_t^i) \quad \text{for} \quad t = 0, 1, 2, ...$$

where K_t^i is capital given to agent i at each t.

Uncertainty There are two states $s_t \in \{H, L\}$ with respect to the productivity A_t^i of each agent i in period t for $t = 1, 2, ...$, which follows a binary Markov process with transition probabilities given by

$$\pi(s|s') = \Pr(s_{t+1} = s' \,|\, s_t = s), \quad s, s' \in \{H, L\}.$$

Denote $s^t = (s_1, \ldots, s_t)$. We define $\pi(s^t)$ by the probability of s^t being realized. The productivity shock is symmetric. In other words, we have either $(A_t^1, A_t^2) = (1+\alpha, 1-\alpha)$ or $(1-\alpha, 1+\alpha)$, so that there is no aggregate productivity shock. At $t = 0$, the probability of $(A_1^1, A_1^2) = (1 + \alpha, 1 - \alpha)$ is $\frac{1}{2}$.

The assumption of the negative correlation is a simple way of generating a situation in which agents are better off ex-ante when they agree on a resource allocation; it not only reduces consumption volatility, but also enlarges the production possibility frontier by allocating more resources to the agent who achieves higher productivity. We assume that the planner allocates capital for each agent after the shock is realized.

2 Optimal Contract with Commitment

We first derive the optimal contract (the first-best allocation) between two agents without considering whether one of the agents can deviate from the contract at any given time. The planner's problem is to allocate consumption $\{(c^1(s^t), c^2(s^t))\}_{t=1}^{\infty}$ and capital $\{(K^1(s^t), K^2(s^t))\}_{t=0}^{\infty}$ to each agent in order to maximize

$$\max\ E_0 \left[\sum_{t=1}^{\infty} \beta^t \Big(u(c_t^1(s^t)) + u(c_t^2(s^t)) \Big) \right] = \sum_{t=1}^{\infty} \sum_{s_t} \beta^t \Big(u(c_t^1(s^t)) + u(c_t^2(s^t)) \Big) \pi(s^t|s^0)$$

subject to the resource constraint for each t and s^t:

$$(RC) \qquad \sum_{i=1,2} \{c^i(s^t) + K^i(s^t)\} = \sum_{i=1,2} \{A_{t-1}^i F(K^i(s^{t-1})) + (1-\delta) K^i(s^{t-1})\}$$

$$K^i(s^t) \geq 0 \quad \forall i$$

$$K(s^t) = K^1(s^t) + K^2(s^t),$$

for given initial total capital $K(s^0)$. β is the discount factor and δ is the depreciation rate of capital.

The first-order conditions imply that,

$$u(c^i(s^t)) = \mu(s^t) \quad \forall i = 1, 2 \tag{2.1}$$

$$A^1(s^t) \frac{\partial F(K^1(s^t))}{\partial K^1(s^t)} = A^2(s^t) \frac{\partial F\big(K(s^t) - K^1(s^t)\big)}{\partial K^1(s^t)} \tag{2.2}$$

$$1 = \beta \sum_{s^{t+1}|s^t} \frac{u'(c(s^{t+1}))}{u'(c(s^t))} \pi(s^{t+1}|s^t) \Big\{ A_t^2 \frac{\partial F\big(K(s^t) - K^1(s^t)\big)}{\partial K(s^t)} + (1-\delta) \Big\}, \tag{2.3}$$

where $\mu(s^t)$ is the Lagrange multiplier.

3 Limited Commitment and the Outside Option

This section considers limited commitment and the outside option value. In particular, we assume each agent can deviate from the optimal contract after the resource is allocated and before the next period shock is realized.

In order to set up the participation constraint for each agent, we first need to pin down the outside option value at the beginning of time t defined by $V_a^i(K^i(s^t))$. It is the value the agent will get from his own planning problem with the initial capital $K^i(s^t)$. We assume that once one of the agents deviates from the optimal contract, he cannot form a contract again. This assumption is rather restrictive but simplifies the analysis considerably.

Note that at the end of time period $t-1$ before he is given consumption $c^t(s^t)$, each agent is assigned $K^i(s^t)$ amount of capital. Then, the outside option value is the solution to the following Cass-Koopmans' problem with the initial condition, $k^i(s^t) = K^i(s^t)$[2]:

$$V_a(K^i(s^t)) = \max \sum_{j=t}^{\infty} \sum_{s^j|s^t} \beta^{j-t} u(c^i(s^j)) \pi(s^j|s^t)$$

subject to

$$c^i(s^j) + k^i(s^{j+1}) = A_j^i F(k^i(s^j)) + (1-\delta) k^i(s^j)$$

$$\forall s^j|s^t, \ \forall j \geq t.$$

Notice that $\pi(s^j|s^t) = \Pr(s^j|s^t)$ for $j = t, t+1, t+2, \cdots$.

The following participation constraint should be satisfied by both agents in order for an agent not to deviate from the optimal contract. That is, for all $i = 1, 2$,

$$(PCi) \quad \sum_{j=t}^{\infty} \sum_{s^j|s^t} \beta^{j-t} u(c^i(s^j)) \, \pi(s^j|s^t) \geq V_a^i(K^i(s^t)), \quad \forall s^t, \ \forall j \geq t.$$

The main focus of this paper is to find out when (PCi) is binding and when it is not.

[2] In order to distinguish the allocation from the optimal contract, we use the lower case letter for the outside option value.

4 Deterministic Case

We first consider a deterministic case in which we can derive the analytic solution for both the value from the optimal contract and the value from the outside option. More specifically, we assume the following economic environment.

Assumption 1.

$$\pi(s^t = H \,|s^{t-1} = L) = \pi(s^t = L \,|s^{t-1} = H) = 1.$$

$$u(c) = \log(c), \quad \delta = 1, \quad A_t F(K_t) = A_t K_t^{\gamma} \quad \text{where} \quad \gamma \in [0, 1).$$

4.1 Uniform convergence

Before characterizing the sustainablity condition, we first specify a benchmark endowment economy for comparison with a production economy. The natural candidate is the case when $\gamma = 0$. More precisely, we consider the benchmark endowment economy as the economy in which each agent receives either $\{1+\alpha, 1-\alpha, ...\}$ or $\{1-\alpha, 1+\alpha, ...\}$ sequence of consumption goods. The Pareto optimal solution for this endowment economy is easily characterized: the consumption of both agents is 1 in each period.

The next proposition shows that both the value function from the optimal contract and the value function from the outside option *uniformly* converges to those of the benchmark endowment economy as γ goes to zero.

Proposition 1. *Let $\alpha \in [0, 1)$ and $\beta \in [0, 1)$ be given. Let K be the compact set of interval. Suppose the interval is sufficiently large so that it includes the initial level of capital and the steady state level. Then, the value function from the optimal contract in a production economy ($\gamma > 0$) uniformly converges to the value function from the optimal contract in an endowment economy ($\gamma = 0$) on K.*

Proof. See the Appendix. □

Notice that as with the argument in Proposition 4.1, it is also straightforward to prove that the value function from the outside option in a production economy ($\gamma > 0$) uniformly converges to the value function from the outside option in an endowment economy ($\gamma = 0$) over K.

In summary, our objective is to investigate how much distortion or improvement of risk-sharing will occur relative to the benchmark endowment economy if we allow capital accumulation with a production technology indexed by a positive γ. In this case, the benchmark endowment economy is defined by the uniform limit of the family of production economies with $\gamma > 0$.

4.2 Characterizing the sustainablity condition

We now analytically characterize the set of economies indexed by (α, β, γ) which can support the optimal contract under limited commitment.

Proposition 2. *Under Assumption 1, the first-best allocation is sustainable if and only if*

$$H(\alpha,\beta,\gamma) := \log \Gamma(\alpha,\gamma) - \frac{(1-\gamma)\{\beta\log(1-\alpha)+\log(1+\alpha)\}}{(1-\beta\gamma)(1+\beta)} - \frac{(1-\beta)\gamma}{1-\beta\gamma}\log(1+\alpha) \geq 0 \tag{4.1}$$

where $\Gamma(\alpha,\gamma) = \left(\frac{(1+\alpha)^{\frac{1}{1-\gamma}}+(1-\alpha)^{\frac{1}{1-\gamma}}}{2}\right)^{1-\gamma}$.

Proof. See the Appendix. □

Note that we define the left-hand-side of the inequality condition (4.1) by $H(\alpha, \beta, \gamma)$, the function of (α, β, γ). Mathematically H is the difference between the value from the optimal contract and the autarky value. To link our results with the previous literature on an endowment economy without commitment, we first investigate the relationship between (α, β) and H.

Corollary 1. *For any given $\alpha \in [0,1)$ and $\gamma \in [0,1)$, there exists a unique $\hat{\beta}$ such that for $\beta > \hat{\beta}$ the first best allocation is sustainable.*

Proof. It is easy to verify, $H(\alpha, 0, \gamma) < 0$, $H(\alpha, 1, \gamma) > 0$, and $\frac{\partial^2 H(\alpha,\beta,\gamma)}{\partial \beta^2} > 0$ for any given $\alpha \in [0,1)$ and $\gamma \in [0,1)$. Therefore, there is a unique $\hat{\beta}$ such that the condition (4.1) is violated if $\beta < \hat{\beta}$ and (4.1) is satisfied if $\beta \geq \hat{\beta}$. □

With the same argument, we can prove the following corollary.

Corollary 2. *For any given $\beta \in (0,1)$ and $\gamma \in [0,1)$, there exists a unique $\hat{\alpha}$ such that for $\alpha > \hat{\alpha}$ the first best allocation is sustainable.*

Corollaries 1 and 2 extend the rather well-known results in the literature into a production economy. As the income fluctuation rises, or as agents value future consumption more, the optimal contract, by which agents can achieve risk sharing, is more likely to be sustainable. This intuition survives for any production technology γ. This confirms the validity of our modeling approach.

4.3 General relationship between the sustainability condition and technology

In this section, we present the general relationship between the sustainability condition and γ. To show general relationship between the sustainability condition and γ, we fix an arbitrary number for β, and present the sustainability condition with respect to (α, γ).[3] Figures 1 $\sim$ Figure 4 describe the set of (α, γ) which can sustain the optimal contract for $\beta = 0.3$, $\beta = 0.5$, $\beta = 0.7$,and $\beta = 0.9$ respectively.[4]

[3]Using the three-dimensional representation is very complicated and not informative since the relation is non-monotonic among three parameters.

[4]In all figures, the green area indicates an economy in which the first-best allocation is sustainable, while the black area is not.

(Insert Figure 1 here)

Figures 1 describes the set of (α, γ) which can sustain the optimal contract for $\beta = 0.3$. For an endowment economy with $(\alpha, \beta) = (0.87, 0.3)$ in which the optimal contract is not satisfied, introducing a production technology with $\gamma = 0.2483$ makes the first-best allocation sustainable. It also indicates that the difference between the value from the optimal contract and the value from the outside option tends to increase as γ becomes higher. For example, when $\alpha = 0.87$, there is a monotonicity in the sense that the optimal contract in fact is sustainable for *all* production technologies γ which is greater than 0.2483.

(Insert Figure 3 here)

Figures 3 describes the set of (α, γ) which can sustain the optimal contract for $\beta = 0.7$. In this figure, an endowment economy indexed by $(\alpha, \beta) = (0.35, 0.7)$ satisfies the sustainability condition. As γ increase from zero, H decreases, and when $\gamma = 0.0311$, the optimal contract becomes unsustainable. When γ becomes 0.7697, however, the optimal contract is once again sustainable. In summary, the relationship between the sustainability condition and γ is non-monotonic.

4.4 Further discussion of the non-monotonicity

We discuss in detail how and why there is non-monotonicity. In particular, we investigate why the optimal contract is sustainable for one technology but not another.

Given (α, β), as γ increases, the difference between the marginal productivity of capital of the two agents increases. This means that capital allocated to the agent receiving a positive shock should also increase to equalize the marginal productivity of capital (MPK) across agents. Since the value from the outside option is strictly increasing in capital as shown in the Appendix, the value from the outside option of the agent with a positive shock increases as γ increases. On the other hand, the increase in the difference between MPKs means that the amount of the rise in the aggregate output by optimally allocating capital across agents also increases. In other words, the value from the optimal contract increases as γ increases. Therefore, the overall effect of γ is ambiguous.

(Insert Figure 4 here)

To understand the mechanism behind this non-monotonicity, let us consider particular examples. We consider an economy indexed by $(\alpha, \beta, \gamma) = (0.1, 0.9, x), x \in [0, 1)$. The sustainability condition for these economies is represented in Figure 4. As shown in Figure 4, the condition (4.1) is satisfied for $\gamma \in [0.96, 1)$ and is otherwise violated. In this example, the first-best allocation is not sustainable for an endowment economy with $\gamma = 0$, but is sustainable in a production economy with a large γ.

(Insert Figure 5 here)

We further investigate the time path for output from the optimal contract and the one from autarky for two particular economies, one satisfying the sustainability condition and the other not. First, we consider an economy indexed by $(\alpha, \beta, \gamma) = (0.1, 0.9, 0.5)$. As shown in Figure 4, the optimal contract is not sustainable in this economy. Figure 5 draws the output path by autarky and by the optimal contract. Even if the outcome fluctuation (and hence the consumption fluctuation) is high in autarky, the agent chooses to deviate from the optimal contract.

(Insert Figure 6 here)

Second, we consider an economy indexed by $(\alpha, \beta, \gamma) = (0.1, 0.9, 0.96)$. As shown in Figure 4, the optimal contract is sustainable in this economy. Figure 6 draws the output path by autarky and by the optimal contract in this economy. This figure shows that the agent does not deviate from the optimal contract although the outcome fluctuation (and hence the consumption fluctuation) is relatively low in autarky. The key difference from the economy indexed by $(\alpha, \beta, \gamma) = (0.1, 0.9, 0.5)$, where the agent with a high productivity deviates from the optimal contract, is that the relative difference between outputs from the optimal contract and outputs from the autarky is very large in the economy indexed by $(\alpha, \beta, \gamma) = (0.1, 0.9, 0.96)$. Figure 5 and Figure 6 clearly illustrate that if gains from efficient resource allocation between agents are large enough, the agent with high productivity is less likely to deviate from the optimal contract even if most of the aggregate capital is allocated to the agent, and the agent is allowed to run away with the allocated capital.

4.5 The sustainability condition near the benchmark endowment economy

We further investigate the relationship between γ and the condition (4.1) near the endowment economy. Before we proceed, note that the optimal contract is sustainable in the benchmark endowment economy if and only if the following condition is satisfied:

$$\beta \log(1-\alpha) + \log(1+\alpha) < 0. \tag{4.2}$$

Figure 7 depicts (α, β) which does not satisfy the condition (4.2).

(Insert Figure 7 here)

Condition (4.2) is derived as the limiting case of (4.1) by $\gamma \to 0$. However, one can equivalently obtain the same condition by directly analyzing the benchmark endowment economy: the condition can be derived by comparing the value from the optimal contract and the value from the autarky in the benchmark endowment economy.

We characterize the set of endowment economies indexed by (α, β) under which the increase in the value from the optimal contract is relatively more than the increase in the value from the outside option once we introduce the production technology with a small γ into the endowment economy. In other words, we derive the set of (α, β) satisfying $\frac{\partial H}{\partial \gamma}\Big|_{\gamma=0} > 0$ as summarized in the following proposition.

Proposition 3. *Let $\alpha \in [0,1)$ and $\beta \in [0,1)$ be given. Suppose we introduce a production technology with $\gamma > 0$ into the endowment economy indexed by (α, β) and allow capital accumulation. Then, for the production economy indexed by $(\alpha, \beta, \hat{\gamma})$ with some small $\hat{\gamma}$, the marginal increase in the value from the optimal contract is relatively more than the marginal increase in the value from the outside option if and only if the following condition is satisfied.*

$$\Big(\frac{1+\alpha}{2} - \frac{(1-\beta)\beta}{1+\beta}\Big)\log(1+\alpha) + \Big(\frac{1-\alpha}{2} + \frac{(1-\beta)\beta}{1+\beta}\Big)\log(1-\alpha) > 0 \tag{4.3}$$

Proof. See the Appendix. □

Because of the smoothness of $H(\alpha, \beta, \cdot)$ with respect to γ, the argument is true for any $\gamma < \hat{\gamma}$ once we find any $\hat{\gamma}$ satisfying the proposition.

(Insert Figure 8 here)

Figure 8 depicts (α, β) which satisfies the condition (4.3). The relationship with respect to α and β is non-linear, but the condition is more likely to be satisfied with a large α and either a large or a small β.

(Insert Figure 9 here)

Figure 9 is the intersection of Figures 7 and 8. As shown by Figure 9, a large portion of those (α, β) that do not satisfy the sustainablity condition in the endowment economy satisfy the condition (4.3). For the endowment economy indexed by (α, β) in the colored area of Figure 9, risk-sharing is improved by introducing a production technology even with an extremely low returns to scale in the sense that the distance between the value from the optimal contract and the value from autarky becomes smaller.

4.6 Further discussion of the sustainability condition

So far, we have demonstrated that introducing a technology into the endowment economy, even a technology that has extremely low returns to scale, can improve risk sharing. We now show that for many endowment economies indexed by (α, β) that do not satisfy the sustainabilty condition, the first-best allocation becomes sustainable once we introduce a production technology with a small returns to scale. In general, the statement holds with any returns to scale higher than the threshold.

We present our results in a two-dimensional space of (α, γ) by fixing β. Figures 10 and Figure 11 describe the economies which satisfy the condition (4.1) but does not satisfy (4.2), meaning the first-best allocation is not sustainable in the benchmark endowment economy but it is sustainable in a production economy.

(Insert Figure 10 here)

The left panel of Figure 10 represents the (α, γ)-plane slice of the three-dimensional space to which (α, β) values satisfy

$$\beta \log(1-\alpha) + \log(1+\alpha) = 0.0001 > 0 \tag{4.4}$$

hence violating condition (4.2). Therefore, the first-best allocation is not sustainable in the endowment economy indexed by these (α, β) values. Notice that there are many combinations of (α, β) which satisfy equation (4.4).

Figure 10 describes how the sustainability condition would change for endowment economies satisfying equation (4.4) if we introduce a production technology with a different returns to scale. The first-best allocation is sustainable in the greenish area of the plane, where

$$H(\alpha, \beta, \gamma) > 0 \quad \text{and} \quad \beta = \frac{0.0001 - \log(1+\alpha)}{\log(1-\alpha)}.$$

Note that $H(\alpha, \beta, \gamma) < 0$ in the black area. For a large value of α, introducing a technology with any returns to scale, except for a very small returns to scale, makes the first-best allocation sustainable. For example, if $\alpha = 0.85$, then γ greater than or equal to 0.0069 satisfies the sustainability condition. On the other hand, for a relatively low α, a fairly high returns to scale is required. For example, if $\alpha = 0.6$, then a technology with $\gamma \geq 0.505$ should be introduced.

(Insert Figure 11 here)

A similar story is true for Figure 11. In this case, relative to Figure 10, the condition (4.2) is further violated to the extent that $\beta \log(1-\alpha)+\log(1+\alpha) = 0.01$, while it is 0.0001 at Figure 10. Then, it is necessary to introduce a technology with a higher returns to scale to make the optimal allocation sustainable. For example, when $\alpha = 0.85$, γ greater than or equal to 0.2289 is required to make the optimal contract sustainable. Similarly, when $\alpha = 0.6$, γ greater than or equal to 0.5976 is required. As the violations of condition (4.2) become deeper, there is a tendency to have to introduce a technology with a higher value of γ to achieve sustainability, but this tendency is not monotonic as we demonstrated before.

5 Stochastic Case

This section investigates the stochastic case. While there is no analytic characterization available as in the deterministic case, we confirm that the baseline intuition presented in the previous section still holds for the stochastic case by using numerical computations. In fact, other than the fact that the sustainability condition is more easily violated when the persistence of the idiosyncratic shock increases, the main intuitions derived in Section 4 hold in a stochastic environment.[5]

Notice that as long as there is no aggregate uncertainty, the planner can achieve perfect risk sharing with commitment even if productivity is random at the individual level. Therefore,

[5]The computational procedure is described in the Appendix.

the value from the first-best allocation in the stochastic environment is the same as the one in the deterministic case. The difference in the stochastic environment is solely driven by the change in the values from the outside option.

Let $\pi(s^t = H \,|s^{t-1} = L) = \pi(s^t = L \,|s^{t-1} = H) = \pi$. As the shock becomes more persistent, that is, as $1-\pi$ approaches one, the autarky value of the agent with a positive shock increases and hence the participation constraint is more easily violated.

(Insert Table 1 here)

To see how the sustainability condition changes as the shock becomes more persistent, we consider the following five benchmark deterministic cases: (1) $(\alpha, \beta, \gamma) = (0.135, 0.9, 0.35)$, (2) $(\alpha, \beta, \gamma) = (0.14, 0.9, 0.35)$, (3) $(\alpha, \beta, \gamma) = (0.15, 0.9, 0.35)$, (4) $(\alpha, \beta, \gamma) = (0.16, 0.9, 0.35)$, (5) $(\alpha, \beta, \gamma) = (0.175, 0.9, 0.35)$. For each benchmark case, we gradually increase the persistence of the idiosyncratic shock, and see whether the first-best allocation is sustainable. The results are shown in Table 1. For all cases, the first-best allocation becomes less sustainable, as the persistence of the shock increases. Moreover, when α is smaller, and hence when the gains from the first-best allocation is smaller (Corollary 2), the sustainability condition is easier to be violated. Table 2 shows the similar results with respect to change in γ. Unlike to Table 1 there is no monotonicity with respect to γ; however, the pattern is similar in the sense that the first-best allocation becomes less sustainable, as the persistence of the shock increases.

(Insert Table 2 here)

Similar to Figure 4, we plot (α, γ) which satisfies the sustainability condition in a stochastic environment. Fixing $\beta = 0.9$, we plot four figures with six different persistence ($1 - \pi = 0, 0.2, 0.3, 0.4, 0.5$). The result rarely changes when the persistence increases from 0 to 0.5. When the persistence further increases, the economy with low α is more likely to violate the sustainability condition, as shown in Table 1. In general, the economy with low γ is also more likely to violate the sustainability condition.

(Insert Figure 12 here)

Tables 1 and 2 and Figure 12 are typical results in the stochastic environment. Overall, as shocks become more persistent, the first-best allocation is less likely to be sustainable since the outside option becomes higher (while the optimal contract provides the same value for both the deterministic and stochastic cases).

6 Conclusion

We study the implications of limited commitment for the first-best allocation in a production economy. To this end, we present a model in which two infinitely-lived agents with the same decreasing returns to scale technology face a negatively correlated productivity shock. Under a parametric assumption, the economy is characterized by the subjective discount factor (β), the extent of productivity shock (α) and the returns to scale of production technology (γ). We derive the analytic condition under which the first-best allocation is sustainable with limited commitment. We show that gains from efficient resource allocation between agents can be so large that they can compensate for the increase in the outside option that arises when capital moves from the less-productive to the more-productive agent.

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Appendix: Proofs, Computational Procedure for Stochastic Cases, Tables and Figures

A Proofs

Value function from the optimal contract

We first derive the analytic solution to the value function from the optimal contract. Let us redefine the production function as

$$Y_t = A_t K_t^{\gamma} = K_t^{\gamma} l_t^{1-\gamma} = \hat{F}(K_t, l_t),$$

where $l_t \in \{(1+\alpha)^{\frac{1}{1-\gamma}}, (1-\alpha)^{\frac{1}{1-\gamma}}\}$. Note that the production function is the same as before with a different notation. Let $\Omega(\alpha, \gamma)$ be $\{(1+\alpha)^{\frac{1}{1-\gamma}} + (1-\alpha)^{\frac{1}{1-\gamma}}\}/2$.

(2.2) implies that

$$\hat{F}_1(K_t^1, l_t^1) = \hat{F}_1(K_t^2, l_t^2).$$

By homogeneity we have $\frac{K_t^1}{l_t^1} = \frac{K_t^2}{l_t^2}$. Let us define $k_t^f = \frac{K_t^1}{l_t^1} = \frac{K_t^2}{l_t^2} = \frac{K_t^1+K_t^2}{l_t^1+l_t^2}$. Notice that $c_t^1 = c_t^2$, $l_t^1 + l_t^2 = 2\Omega$ and $k_t^f = \frac{K}{2\Omega}$. The resource constraint collapses to

$$c_t + k_{t+1}^f \Omega(\alpha, \gamma) = f(k_t^f)\Omega(\alpha, \gamma), \tag{A.1}$$

where $f(k) = \hat{F}(k, 1) = k^{\gamma}$. Hence, by the equal treatment condition the planner's problem is to maximize $\sum_{t=0}^{t} \beta^t u(c_t)$ subject to (A.1). This is the same as the classical Cass-Koopmans growth model. Therefore, the solution with full enforcement is easily given by

$$c_t^f = (1-\beta\gamma)\Omega(\alpha,\gamma) f(k_t^f) \quad \text{and} \quad k_{t+1}^f = \beta\gamma f(k_t^f).$$

Recursively, we have $k_{t+j}^f = (\beta\gamma)^{\frac{1-\gamma^j}{1-\gamma}}(k_t^f)^{\gamma^j}$. Therefore, the value at given capital k_t^f is

$$\begin{aligned}
V(k_t^f) &= \sum_{j=0}^{\infty} \beta^j \log(c_{t+j}) \\
&= \sum_{j=0}^{\infty} \beta^j \log\Big((1-\beta\gamma)\Omega(\alpha,\gamma) f(k_{t+j}^f)\Big) \\
&= \frac{\log\Big((1-\beta\gamma)\Omega(\alpha,\gamma)\Big)}{1-\beta} + \gamma \sum_{j=0}^{\infty} \beta^j \log(k_{t+j}^f) \\
&= \frac{\log\Big((1-\beta\gamma)\Omega(\alpha,\gamma)\Big)}{1-\beta} + \frac{\gamma\log(\beta\gamma)}{1-\gamma}\left(\frac{1}{1-\beta} - \frac{1}{1-\gamma\beta}\right) + \frac{\gamma \log k_t^f}{1-\beta\gamma} \\
&= \frac{1}{1-\beta}\left[\log\Big((1-\beta\gamma)\Omega(\alpha,\gamma)\Big) + \frac{\beta\gamma\log(\beta\gamma)}{1-\beta\gamma}\right] + \frac{\gamma\log k_t^f}{1-\beta\gamma}.
\end{aligned}$$

Value function from the outside option

Suppose an agent is given K unit of capital and wants to deviate. Then, the autarky values $(V_a(K,H), V_a(K,L))$ are given by the following Bellman equation.

$$\begin{aligned}
V_a(K,L) &= \max_{K'=K'(L)} \log\Big((1-\alpha)K^\gamma - K'\Big) + \beta V_a(K',H) \\
V_a(K,H) &= \max_{K'=K'(H)} \log\Big((1+\alpha)K^\gamma - K'\Big) + \beta V_a(K',L), \qquad \text{(A.2)}
\end{aligned}$$

where $K'(s)$ is investment given the current state $s \in \{H, L\}$. Note that in the deterministic case we consider, $\pi(H|H) = \pi(L|L) = 0$. We proceed with the following guess and verify the argument. Guess that

$$V_a(K,L) = a_L \log K + x \quad \text{and} \quad V_a(K,H) = a_H \log K + y.$$

The first order condition and the envelop theorem give $a_H = a_L = \frac{\gamma}{1-\beta\gamma}$ and

$$K'(L) = \beta\gamma F(K, 1-\alpha) \quad \text{and} \quad K'(H) = \beta\gamma F(K, 1+\alpha). \qquad \text{(A.3)}$$

Then, putting (A.3) into the Bellman equation (A.2), we can match the coefficients with respect to x and y to get the following linear equation:

$$\begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma) + \frac{1}{1-\beta\gamma}\log(1-\alpha) \\ \log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma) + \frac{1}{1-\beta\gamma}\log(1+\alpha) \end{bmatrix}.$$

Then, we get

$$x = \frac{1}{1-\beta}\left[\log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma)\right] + \frac{\log(1-\alpha)+\beta\log(1+\alpha)}{(1-\beta\gamma)(1-\beta)(1+\beta)}$$
$$y = \frac{1}{1-\beta}\left[\log(1-\beta\gamma) + \frac{\beta\gamma}{1-\beta\gamma}\log(\beta\gamma)\right] + \frac{\beta\log(1-\alpha)+\log(1+\alpha)}{(1-\beta\gamma)(1-\beta)(1+\beta)}$$

which verifies the guess.

Proof of Proposition 1

Given the policy function $k_{t+1}^f = \beta\gamma f(k_t^f)$ which is derived above, the sequence of $\{k_t\}$ with any initial capital k_0 converges the steady state $k^* = (\beta\gamma)^{\frac{1}{1-\gamma}}$. Therefore, it is sufficient to consider the uniform convergent over $[k_0, k^*]$ or $[k^*, k_0]$. Without loss of generality, we consider the case where $k_0 < k^*$. Note that each agent can consume one unit of consumption for all periods from the optimal contact in the endowment economy. Therefore, the value function from the optimal contract in the endowment economy is zero with log utility. To show the uniform convergence, it is sufficient to find γ_ϵ such that $|V(k,\gamma) < \epsilon|$ for $\gamma < \gamma_\epsilon$ for all $k \in [k_0, k^*]$.

Denote $V(k,\gamma) = G(\gamma) + \frac{\gamma \log k}{1-\beta\gamma}$. We first consider the pointwise convergence at k^*. Since $\lim_{\gamma\to 0} G(\gamma) = 0$, we can find γ_ϵ^1 such that $|G(\gamma)| < \frac{\epsilon}{2}$ is satisfied for $\gamma < \gamma_\epsilon^1$. If we take $\gamma_\epsilon^2(k^*)$ as $\frac{\epsilon}{2\log k^* + \epsilon\beta}$, then $|\frac{\gamma \log k^*}{1-\beta\gamma}| < \frac{\epsilon}{2}$ for $\gamma < \gamma_\epsilon^2(k^*)$. Define $\gamma_\epsilon(k^*)$ as $\min\{\gamma_\epsilon^1, \gamma_\epsilon^2(k^*)\}$. Then $|V(k^*,\gamma)| < \epsilon$ for any $\gamma < \gamma_\epsilon(k^*)$. Therefore, $V(k,\gamma)$ converges to zero pointwise at k^*. In fact, if we take $\gamma_\epsilon = \gamma_\epsilon(k^*)$, then $|V(k,\gamma)| < \epsilon$ for any $\gamma < \gamma_\epsilon$ for all $k \in [k_0, k^*]$ since $\log k < \log k^*$ for any $k \in [k_0, k^*)$. This completes the proof.

Proof of Proposition 2

Notice that $\pi_H = \pi_L = 0$ implies $V_a(K, L) < V_a(K, H)$ where $V_a(K, s)$ is the outside option value given capital K with state $s \in \{H, L\}$. An agent will choose autarky only when he receives a large endowment. Therefore, he will never deviate if the following is satisfied:

$$V(\frac{K}{2\Omega}) \geq V_a(\frac{K(1+\alpha)^{\frac{1}{1-\gamma}}}{2\Omega}, H).$$

The above inequality is equivalent to (4.1), which completes the proof.

Proof of Proposition 3

The partial derivative of H with respect to γ yields

$$\begin{aligned}\frac{\partial H}{\partial \gamma} = &- \log\Big(\frac{(1+\alpha)^{\frac{1}{1-\gamma}} + (1-\alpha)^{\frac{1}{1-\gamma}}}{2}\Big) \\ &+ \frac{1}{1-\gamma}\Big(\frac{(1+\alpha)^{\frac{1}{1-\gamma}}\log(1+\alpha) + (1-\alpha)^{\frac{1}{1-\gamma}}\log(1-\alpha)}{(1+\alpha)^{\frac{1}{1-\gamma}} + (1-\alpha)^{\frac{1}{1-\gamma}}}\Big) \\ &+ \frac{1-\beta}{(1-\beta\gamma)^2}\Big(\frac{\beta\{\log(1-\alpha) - \log(1+\alpha)\}}{1+\beta}\Big).\end{aligned}$$

Therefore, $\frac{\partial H}{\partial \gamma}\Big|_{\gamma=0} > 0$ generates the equation (4.3).

B Computational Procedure for Stochastic Case

1. Derive the value function from the first-best allocation.

   $$V(K) = max_{\{c,K^1,K^2,K'\}} \log(c) + \beta V(K')$$

   subject to

   $$c = \frac{1}{2}\Big\{(1+\alpha)F(K^1) + (1-\alpha)F(K^2) + (1-\delta)K - K'\Big\}$$

   for $K = K^1 + K^2$.

2. Derive the value function for autarky.

$$V_a(k,s) = max_{\{c,k\}} \log(c) + \beta E\left[V_a(k',s')|s\right]$$

subject to

$$c = A(s)F(k) + (1-\delta)k - k'.$$

3. Given K, K^i is given by the policy function derived in Step 1, $K^i = K^i(K)$. The first-best allocation is sustainable if and only if

$$V(K) \geq V_a(K^1(K), H).$$

C Tables and Figures

Table 1: Persistence of Shock and the Sustainability Condition

		Persistence $(1-\pi)$					
		0	0.1	0.2	0.3	0.4	0.5
	0.135	O	X	X	X	X	X
	0.14	O	O	X	X	X	X
α	0.15	O	O	O	X	X	X
	0.16	O	O	O	O	X	X
	0.175	O	O	O	O	O	X

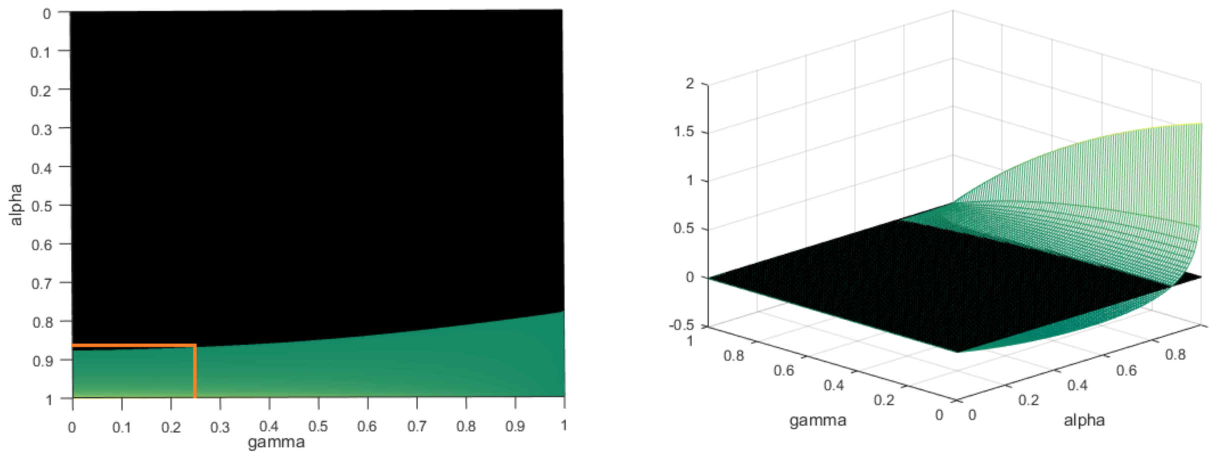
NOTE: For all simulations, β and γ are fixed as 0.9 and 0.35, respectively. $1-\pi$ represents the persistence of the idiosyncratic shock. $1-\pi=0$ corresponds to the benchmark deterministic case. O and X indicate the first-best allocation being sustainable and not being sustainable, respectively.

Table 2: Persistence of Shock and the Sustainability Condition

		Persistence ($1-\pi$)					
		0	0.1	0.2	0.3	0.4	0.5
	0.1	O	O	O	X	X	X
	0.2	O	O	O	X	X	X
	0.3	O	O	O	X	X	X
	0.4	O	O	O	X	X	X
γ	0.5	O	O	X	X	X	X
	0.6	O	O	X	X	X	X
	0.7	O	O	O	X	X	X
	0.8	O	O	O	O	O	X
	0.9	O	O	O	O	O	O

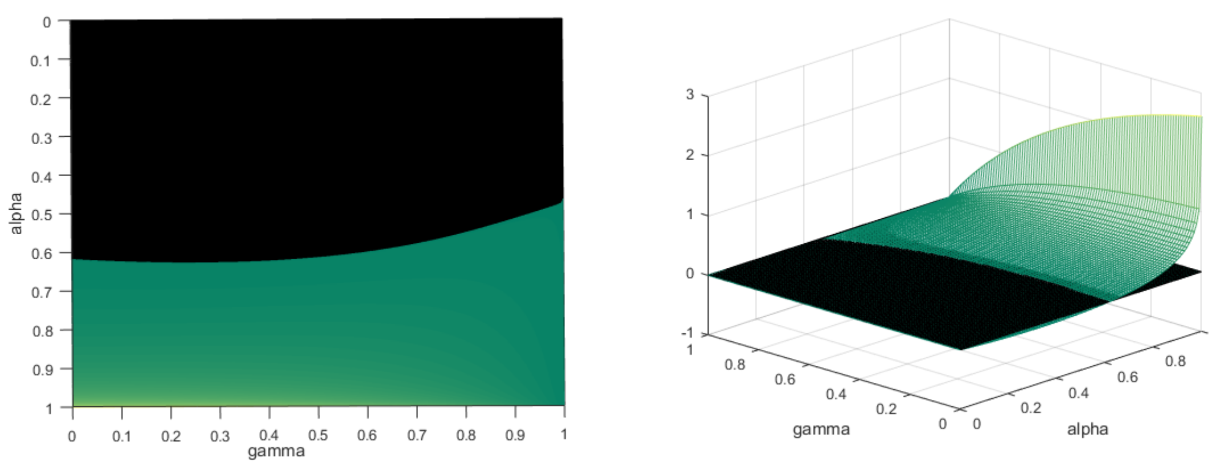
NOTE: For all simulations, β and α are fixed as 0.9 and 0.1, respectively. $1-\pi$ represents the persistence of the idiosyncratic shock. $1-\pi=0$ corresponds to the benchmark deterministic case. O and X indicate the first-best allocation being sustainable and not being sustainable, respectively.

Figure 1: Left: (α, γ) satisfying $H(\alpha, \beta = 0.3, \gamma) > 0$; Right: $H(\alpha, \beta = 0.3, \gamma)$



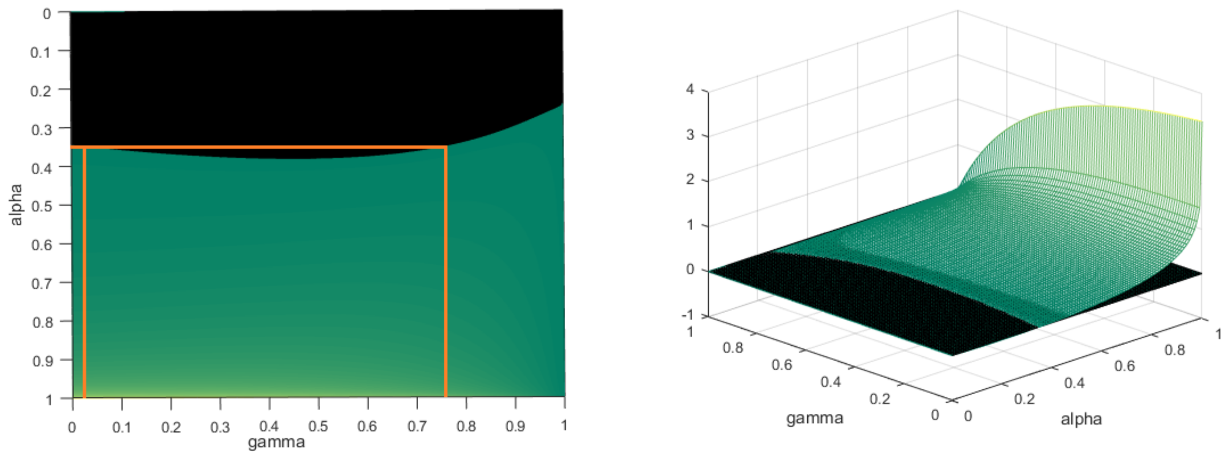
NOTE: Left: The green area indicates an economy in which the first-best allocation is sustainable; Right: The green meshed line indicates the value for $H(\alpha, \beta = 0.3, \gamma)$ and the black surface indicates $H = 0$.

Figure 2: Left: (α, γ) satisfying $H(\alpha, \beta = 0.5, \gamma) > 0$; Right: $H(\alpha, \beta = 0.5, \gamma)$



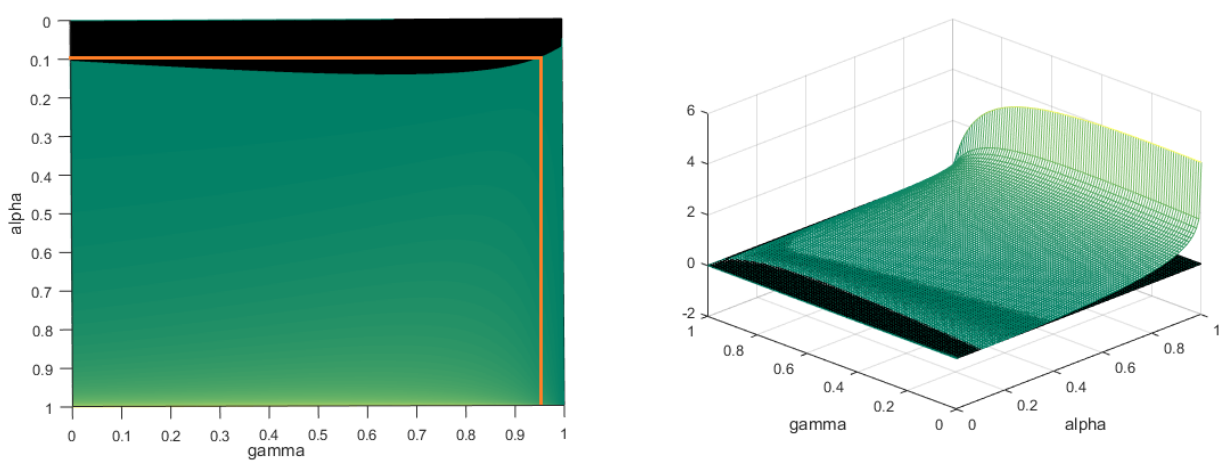
NOTE: Left: The green area indicates an economy in which the first-best allocation is sustainable; Right: The green meshed line indicates the value for $H(\alpha, \beta = 0.5, \gamma)$ and the black surface indicates $H = 0$.

Figure 3: Left: (α, γ) satisfying $H(\alpha, \beta = 0.7, \gamma) > 0$; Right: $H(\alpha, \beta = 0.7, \gamma)$



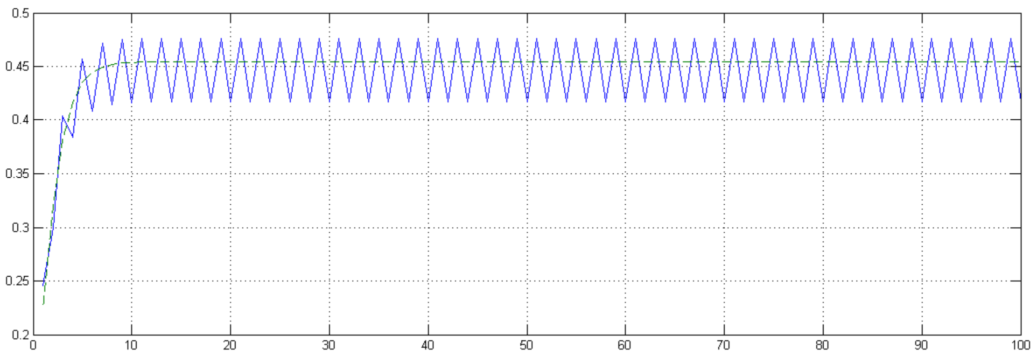
NOTE: Left: The green area indicates an economy in which the first-best allocation is sustainable; Right: The green meshed line indicates the value for $H(\alpha, \beta = 0.7, \gamma)$ and the black surface indicates $H = 0$.

Figure 4: Left: (α, γ) satisfying $H(\alpha, \beta = 0.9, \gamma) > 0$; Right: $H(\alpha, \beta = 0.9, \gamma)$



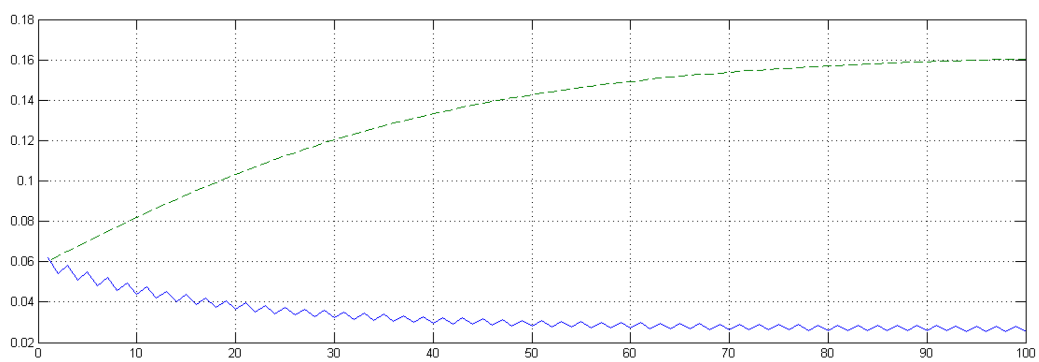
NOTE: Left: The green area indicates an economy in which the first-best allocation is sustainable; Right: The green meshed line indicates the value for $H(\alpha, \beta = 0.9, \gamma)$ and the black surface indicates $H = 0$.

Figure 5: Time path for output *per capita* ($\alpha = 0.1, \beta = 0.9, \gamma = 0.5, K_0 = 0.1$)



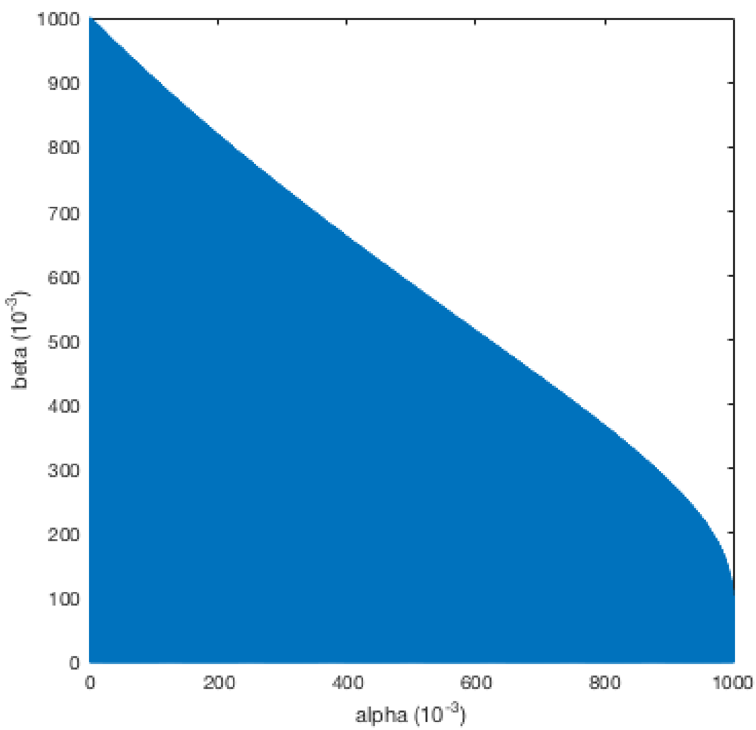
NOTE: The blue line represents the outside option value with $k_0 = K_0/2$. The dotted green line represents the value from the optimal contract.

Figure 6: Time path for output *per capita* ($\alpha = 0.1, \beta = 0.9, \gamma = 0.96, K_0 = 0.1$)



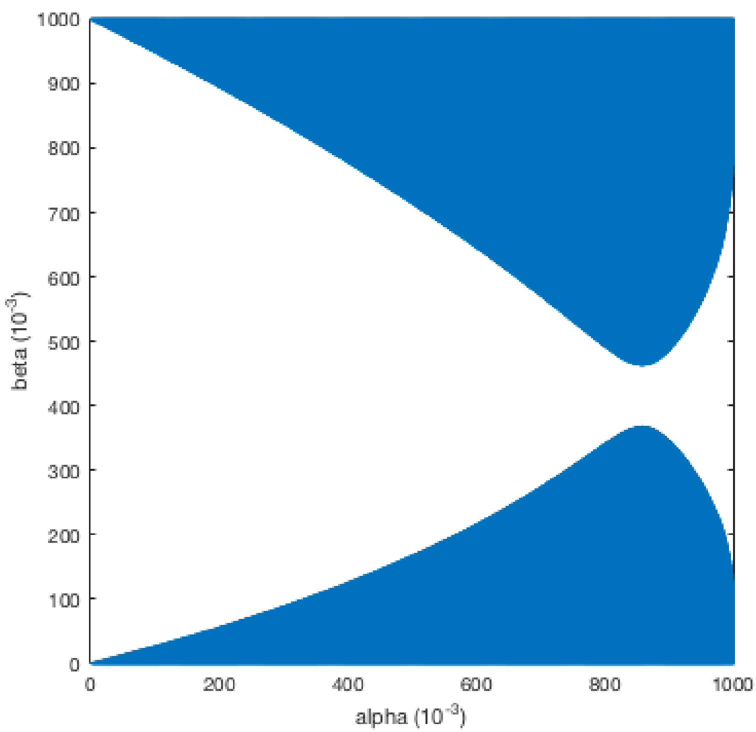
NOTE: The blue line represents the outside option value with $k_0 = K_0/2$. The dotted green line represents the value from the optimal contract.

Figure 7: (α, β) not satisfying the sustainability condition in the endowment economy



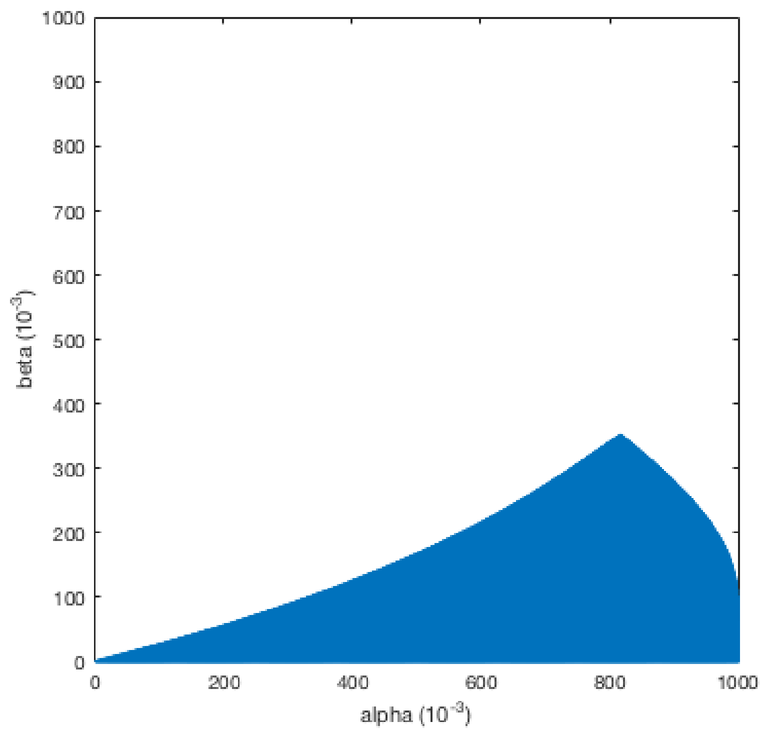
NOTE: (α, β) with color indicates an endowment economy not satisfying the sustainability condition.

Figure 8: (α, β) generating a positive value for the condition (4.3)



NOTE: (α, β) with color indicates the region in which (4.3) is satisfied.

Figure 9: (α, β) not satisfying the sustainability condition in the endowment economy and generating a positive value for the condition (4.3)



NOTE: (α, β) with color indicates an endowment economy in which the value from the optimal contract increases faster than the value from the outside option once we introduce the production technology with γ which is marginally greater than zero, and at the same time, not satisfying the sustainability condition in the endowment economy.

Figure 10: Left: (α, γ)-plane satisfying $\beta = \frac{0.0001-\log(1+\alpha)}{\log(1-\alpha)}$. In this case, $H(\alpha, \beta = \frac{0.0001-\log(1+\alpha)}{\log(1-\alpha)}, \gamma) > 0$ for the greenish area while $H(\alpha, \beta = \frac{0.0001-\log(1+\alpha)}{\log(1-\alpha)}, \gamma) < 0$ for the black area; Right: The greenish surface plots $H(\alpha, \beta = \frac{0.0001-\log(1+\alpha)}{\log(1-\alpha)}, \gamma)$ values in three-dimension.

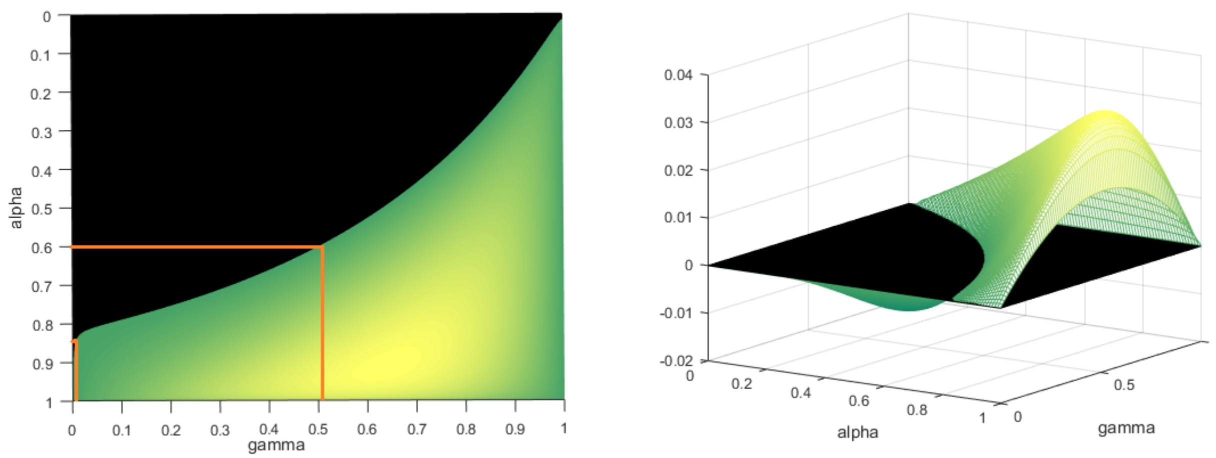


Figure 11: Left: (α, γ)-plane satisfying $\beta = \frac{0.01-\log(1+\alpha)}{\log(1-\alpha)}$. In this case, $H(\alpha, \beta = \frac{0.01-\log(1+\alpha)}{\log(1-\alpha)}, \gamma) > 0$ for the greenish area while $H(\alpha, \beta = \frac{0.01-\log(1+\alpha)}{\log(1-\alpha)}, \gamma) < 0$ for the black area; Right: The greenish surface plots $H(\alpha, \beta = \frac{0.01-\log(1+\alpha)}{\log(1-\alpha)}, \gamma)$ values in three-dimension.

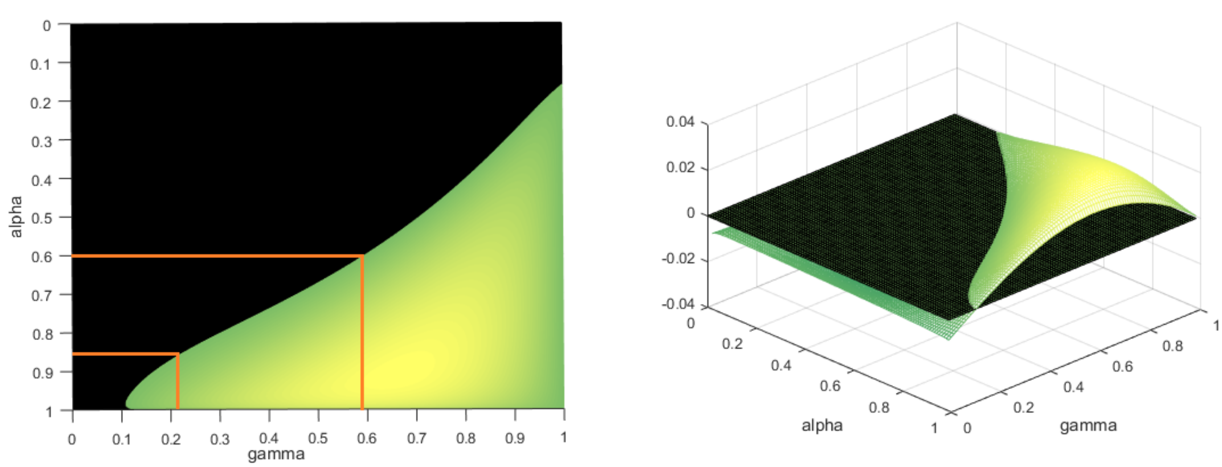
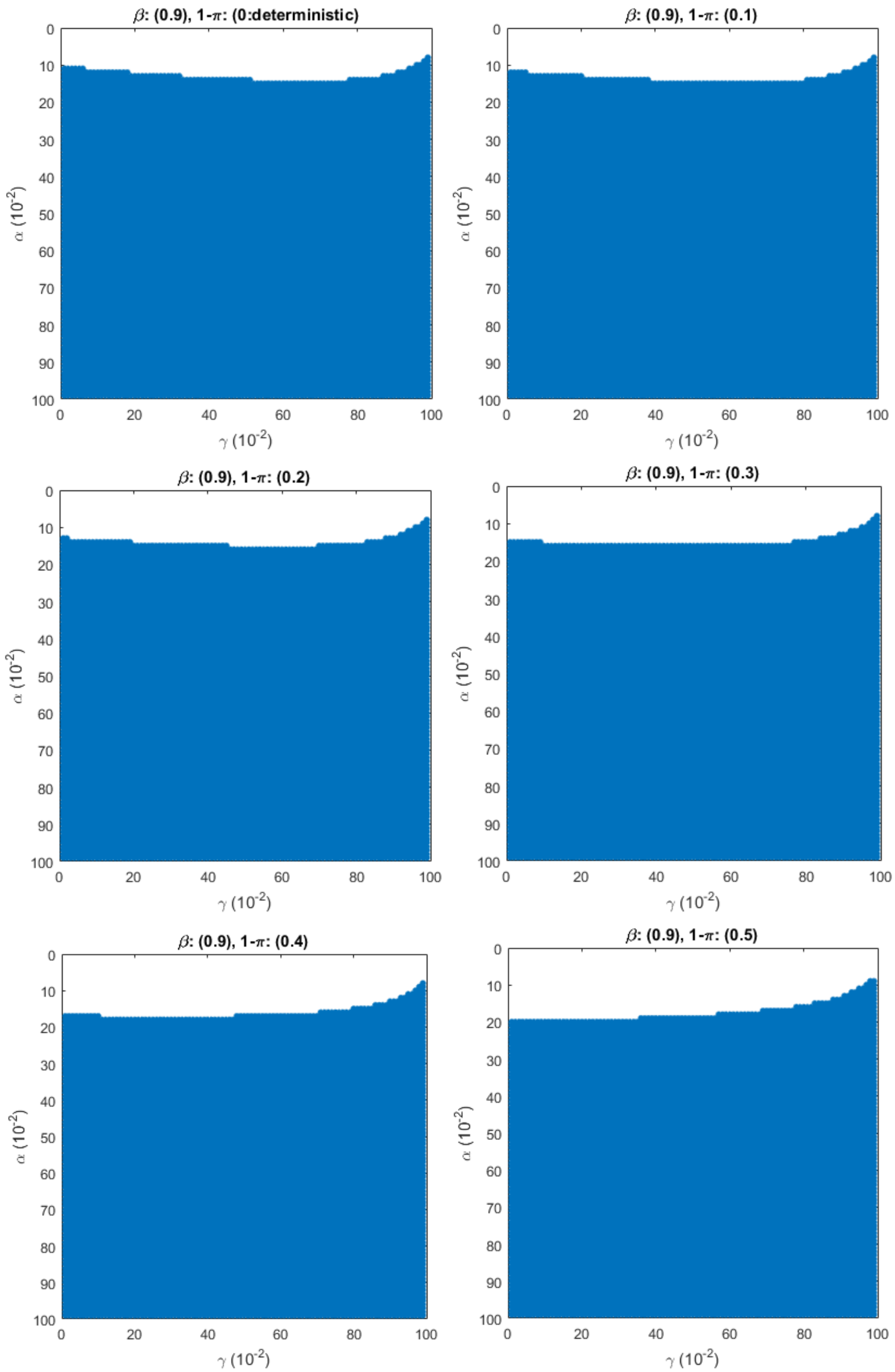


Figure 12: (α, γ) satisfying the sustainability condition: Top Left: $(\beta, 1-\pi) = (0.9, 0)$, Top Right: $(\beta, 1-\pi) = (0.9, 0.1)$, Middle Left: $(\beta, 1-\pi) = (0.9, 0.2)$, Middle Right: $(\beta, 1-\pi) = (0.9, 0.3)$, Bottom Left: $(\beta, 1-\pi) = (0.9, 0.4)$, Bottom Right: $(\beta, 1-\pi) = (0.9, 0.5)$



NOTE: (α, γ) with color indicates the economy satisfying the sustainability condition. The domain for α and γ is $\{0.01, 0.02, ..., 0.99\}$.